Integration of vector hydrodynamical partial differential equations over octonions.

Ludkovsky S.V.

29 December 2010

Abstract

New technique of integration of certain types of partial differential equations is developed. For this purpose non-commutative integration over Cayley-Dickson algebras is used. Applications to non-linear vector partial differential equations of Korteweg-de-Vries and Kadomtzev-Petviashvili types and describing non-isothermal flows of incompressible Newtonian liquids are given.

[1]

1 Introduction.

This article is devoted to a method of non-commutative integration of systems of partial differential equations or a partial differential equation written in a vector form over Cayley-Dickson algebras. This technique is applied to non-linear generalized Korteweg-de-Vries and Kadomtzev-Petviashvili and also describing non-isothermal flows of incompressible Newtonian liquids partial differential equations written in vector forms with large number of variables. Dirac had used complexified quaternions to solve Klein-Gordon's hyperbolic partial differential equation, which is used in quantum mechanics. This approach is generalized in this paper for non-linear partial differential equations over the Cayley-Dickson algebras.

The method based on **R**-linear integral equations was previously used for non-linear partial differential equations with a real time variable t and real space variables x, y [1, 17]. Using the non-commutative line integration over

[1]key words and phrases: hypercomplex, octonions, non-linear partial differential equation, non-commutative integration

Mathematics Subject Classification 2000: 30G35, 32W50, 35G20

the Cayley-Dickson algebras it is spread in this paper for variables x, y in the Euclidean space $\mathbf{R}^n$, $n \in \mathbf{N}$.

Korteweg-de-Vries equations describe a motion of shallow waters, for example, in channels. Kadomtzev-Petviashvili and modified Korteweg-de-Vries equations are used in modeling of physical processes such as non-stationary spread of waves in a material with dispersion, magneto-hydrodynamical waves in a non-collision plasma, heat conductivity of a lattice of anharmonic oscillators. Moreover, partial differential equations describing stationary non-isothermal flows of incompressible Newtonian liquids are very important as well. This method is also applicable to a resonance interactions in a non-linear material, for example, in the non-linear optics.

In this article previous results of the author on functions of Cayley-Dickson variable and non-commutative line integrals over Cayley-Dickson algebras are used [11, 9, 10, 16].

Henceforward, the notations of previous papers [9, 16] and the book [11] are used.

2 Integration of partial differential equations over Cayley-Dickson algebras.

1.1. Notations and Definitions. Let $\mathcal{A}_r$ denote the real Cayley-Dickson algebra with generators $i_0, ..., i_{2^r-1}$ such that $i_0 = 1$, $i_j^2 = -1$ for each $j \ge 1$, $i_j i_k = -i_k i_j$ for each $j \ne k \ge 1$. It is supposed further, that a domain U in $\mathcal{A}_r^m$ has the property that

$(D1)$ each projection $\mathbf{p}_j(U) =: U_j$ is $(2^r - 1)$-connected;

$(D2)$ $\pi_{\mathsf{s},\mathsf{p},\mathsf{t}}(U_j)$ is simply connected in $\mathbf{C}$ for each $k = 0, 1, ..., 2^{r-1}$, $\mathsf{s} = i_{2k}$, $\mathsf{p} = i_{2k+1}$, $\mathsf{t} \in \mathcal{A}_{r,\mathsf{s},\mathsf{p}}$ and $\mathsf{u} \in \mathbf{C}_{\mathsf{s},\mathsf{p}}$, for which there exists $z = \mathsf{u} + \mathsf{t} \in U_j$, where $e_j = (0, ..., 0, 1, 0, ..., 0) \in \mathcal{A}_r^m$ is the vector with 1 on the j-th place, $\mathbf{p}_j(z) = {}^j z$ for each $z \in \mathcal{A}_r^m$, $z = \sum_{j=1}^m {}^j z e_j$, ${}^j z \in \mathcal{A}_r$ for each $j = 1, ..., m$, $m \in \mathbf{N} := \{1, 2, 3, ...\}$, $\pi_{\mathsf{s},\mathsf{p},\mathsf{t}}(V) := \{\mathsf{u} : z \in V, z = \sum_{\mathsf{v} \in \mathbf{b}} w_{\mathsf{v}} \mathsf{v}, \mathsf{u} = w_{\mathsf{s}} \mathsf{s} + w_{\mathsf{p}} \mathsf{p}\}$ for a domain V in $\mathcal{A}_r$ for each $\mathsf{s} \ne \mathsf{p} \in \mathbf{b}$, where $\mathsf{t} := \sum_{\mathsf{v} \in \mathbf{b} \setminus \{\mathsf{s},\mathsf{p}\}} w_{\mathsf{v}} \mathsf{v} \in \mathcal{A}_{r,\mathsf{s},\mathsf{p}} := \{z \in \mathcal{A}_r : z = \sum_{\mathsf{v} \in \mathbf{b}} w_{\mathsf{v}} \mathsf{v}, w_{\mathsf{s}} = w_{\mathsf{p}} = 0, w_{\mathsf{v}} \in \mathbf{R} \ \forall \mathsf{v} \in \mathbf{b}\}$, where $\mathbf{b} := \{i_0, i_1, ..., i_{2^r-1}\}$ is the family of standard generators of the Cayley-Dickson algebra $\mathcal{A}_r$. Frequently we take $m = 1$. Henceforward, we consider a domain U satisfying Conditions $(D1, D2)$ if any other is not outlined.

The Cayley-Dickson algebra $\mathcal{A}_{r+1}$ is formed from the algebra $\mathcal{A}_r$ with the help of the doubling procedure by generator i_{2^r}, in particular, $\mathcal{A}_0 := \mathbf{R}$ is the real field, $\mathcal{A}_1 = \mathbf{C}$ coincides with the field of complex numbers, $\mathcal{A}_2 = \mathbf{H}$

is the skew field of quaternions, $\mathcal{A}_3$ is the algebra of octonions, $\mathcal{A}_4$ is the algebra of sedenions. The skew field of quaternions is associative, and the algebra of octonions is alternative. The multiplication of arbitrary octonions ξ, η satisfies equations $(1,2)$ below:

(1) $(\xi\eta)\eta = \xi(\eta\eta)$,

(2) $\xi(\xi\eta) = (\xi\xi)\eta$,

that forms the alternative system. The algebra $\mathcal{A}_r$ is power associative, that is, $z^{n+m} = z^n z^m$ for each $n, m \in \mathbf{N}$ and $z \in \mathcal{A}_r$, it is non-associative and non-alternative for each $r \geq 4$.

The Cayley-Dickson algebras are $*$-algebras, that is, there is a real-linear mapping $\mathcal{A}_r \ni a \mapsto a^* \in \mathcal{A}_r$ such that

(2) $a^{**} = a$,

(3) $(ab)^* = b^* a^*$ for each $a, b \in \mathcal{A}_r$. Then they are nicely normed, that is,

(4) $a + a^* =: 2Re(a) \in \mathbf{R}$ and

(5) $aa^* = a^* a > 0$ for each $0 \neq a \in \mathcal{A}_r$. The norm in it is defined by the equation:

(6) $|a|^2 := aa^*$.

We also denote a^* by $\tilde{a}$. Each non-zero Cayley-Dickson number $0 \neq a \in \mathcal{A}_r$ has the multiplicative inverse given by $a^{-1} = a^*/|a|^2$.

The doubling procedure is as follows. Each $z \in \mathcal{A}_{r+1}$ is written in the form $z = a + b\mathbf{l}$, where $\mathbf{l}^2 = -1$, $\mathbf{l} \notin \mathcal{A}_r$, $a, b \in \mathcal{A}_r$. The addition is componentwise. The conjugate of a Cayley-Dickson number z is prescribed by the formula:

(7) $z^* := a^* - b\mathbf{l}$.

The multiplication is given by Equation

(8) $(\alpha + \beta\mathbf{l})(\gamma + \delta\mathbf{l}) = (\alpha\gamma - \tilde{\delta}\beta) + (\delta\alpha + \beta\tilde{\gamma})\mathbf{l}$

for each $\alpha,\ \beta,\ \gamma,\ \ \delta \in \mathcal{A}_r,\ \ \xi := \alpha + \beta\mathbf{l} \in \mathcal{A}_{r+1},\ \ \eta := \gamma + \delta\mathbf{l} \in \mathcal{A}_{r+1}$.

The basis of $\mathcal{A}_{r+1}$ over $\mathbf{R}$ is denoted by $\mathbf{b}_{r+1} := \mathbf{b} := \{1, i_1, ..., i_{2^{r+1}-1}\}$, where $i_s^2 = -1$ for each $1 \leq s \leq 2^{r+1} - 1$, $i_{2^r} := \mathbf{l}$ is the additional element of the doubling procedure of $\mathcal{A}_{r+1}$ from $\mathcal{A}_r$, choose $i_{2^r+m} = i_m\mathbf{l}$ for each $m = 1, ..., 2^r - 1$, $i_0 := 1$.

1.2. Operators. An $\mathbf{R}$ linear space X which is also left and right $\mathcal{A}_r$ module will be called an $\mathcal{A}_r$ vector space. We present X as the direct sum $X = X_0 i_0 \oplus ... \oplus X_{2^r-1} i_{2^r-1}$, where X_0,...,X_{2^r-1} are pairwise isomorphic real linear spaces. Certainly, for $r = 2$ this module is associative: $(xa)b = x(ab)$ and $(ab)x = a(bx)$ for all $x \in X$ and $a, b \in \mathbf{H}$, since the quaternion skew field $\mathcal{A}_2 = \mathbf{H}$ is associative. This module is alternative for $r = 3$: $(xa)a = x(a^2)$ and $(a^2)x = a(ax)$ for all $x \in X$ and $a \in \mathbf{O}$, since the octonion algebra $\mathbf{O} = \mathcal{A}_3$ is alternative.

Let X and Y be two $\mathbf{R}$ linear normed spaces which are also left and right

$\mathcal{A}_r$ modules, where $1 \le r$, such that $0 \le \|ax\|_X = |a|\|x\|_X = \|xa\|_X$ and $\|x+y\|_X \le \|x\|_X + \|y\|_X$ for all $x, y \in X$ and $a \in \mathcal{A}_r$. Such spaces X and Y will be called $\mathcal{A}_r$ normed spaces.

We say that an $\mathcal{A}_r$ vector space X is supplied with an $\mathcal{A}_r$ valued scalar product, if

$(f,g) = \sum_{j,k=0}^{2^r-1} (f_j, g_k) i_j^* i_k,$

where $f = f_0 i_0 + ... + f_{2^r-1} i_{2^r-1}$, $f, g \in X$, $f_j, g_j \in X_j$, each X_j is a real linear space with a real valued scalar product, $(X_j, (*,*))$ is real linear isomorphic with $(X_k, (*,*))$ and $(f_j, g_k) \in \mathbf{R}$ for each j, k. The scalar product induces the norm: $\|f\| := \sqrt{(f,f)}$.

An $\mathcal{A}_r$ normed space or an $\mathcal{A}_r$ vector space with $\mathcal{A}_r$ scalar product complete relative to its norm will be called an $\mathcal{A}_r$ Banach space or an $\mathcal{A}_r$ Hilbert space respectively.

We put $X^{\otimes k} := X \otimes_{\mathbf{R}} ... \otimes_{\mathbf{R}} X$ to be the k times ordered tensor product over $\mathbf{R}$ of X. By $L_{q,k}(X^{\otimes k}, Y)$ we denote a family of all continuous k times $\mathbf{R}$ poly-linear and $\mathcal{A}_r$ additive operators from $X^{\otimes k}$ into Y. If X and Y are normed $\mathcal{A}_r$ spaces and Y is complete relative to its norm, then $L_{q,k}(X^{\otimes k}, Y)$ is also a normed $\mathbf{R}$ linear and left and right $\mathcal{A}_r$ module complete relative to its norm. In particular, $L_{q,1}(X,Y)$ is denoted also by $L_q(X,Y)$.

If $A \in L_q(X,Y)$ and $A(xb) = (Ax)b$ or $A(bx) = b(Ax)$ for each $x \in X_0$ and $b \in \mathcal{A}_r$, then an operator A we call right or left $\mathcal{A}_r$-linear respectively.

An $\mathbf{R}$ linear space of left (or right) k times $\mathcal{A}_r$ poly-linear operators is denoted by $L_{l,k}(X^{\otimes k}, Y)$ (or $L_{r,k}(X^{\otimes k}, Y)$ respectively).

An $\mathbf{R}$-linear operator $A : X \to X$ will be called right or left strongly $\mathcal{A}_r$ linear if

$A(xb) = (Ax)b$ or

$A(bx) = b(Ax)$ for each $x \in X$ and $b \in \mathcal{A}_r$ correspondingly.

1.2.1. Examples. Each right $\mathcal{A}_r$ linear operator $A : X \to X$ such that $Ax \in X_0$ for each $x \in X_0$ is strongly right $\mathcal{A}_r$ linear, since

$A(xb) = \sum_{j,k=0}^{2^r-1} A(x_j i_j b_k i_k) = \sum_{j,k=0}^{2^r-1} (Ax_j) i_j i_k b_k$

$= \sum_{j,k=0}^{2^r-1} (Ax_j i_j) i_k b_k = (Ax)b,$

since $Ax_j \in X_j$ up to an $\mathbf{R}$ linear isomorphism for each $x = x_0 i_0 + ... + x_{2^r-1} i_{2^r-1} \in X$ and $b = b_0 i_0 + ... + b_{2^r-1} i_{2^r-1} \in \mathcal{A}_r$ with $x_j \in X_j$ and $b_j \in \mathbf{R}$ for each $j = 0, ..., 2^r - 1$.

Particularly, if $A \in L_r(X,Y)$ is right $\mathcal{A}_2$ linear with X and Y over $\mathcal{A}_2$, i.e. for X and Y over the quaternion skew field $\mathbf{H}$, then A is strongly right $\mathbf{H}$ linear, since the quaternion skew field is associative and $x(ab) = (xa)b$ for each $x \in X$ and $a, b \in \mathbf{H}$.

1.2.2. Lemma. *If an invertible operator A is either right $\mathcal{A}_r$ linear or*

right strongly $\mathcal{A}_r$ linear, then A^{-1} is such also.

Proof. This follows from the equalities:

$(A^{-1}y)b = (A^{-1}Ax)b = xb = A^{-1}(A(xb)) = A^{-1}((Ax)b) = A^{-1}(yb)$,
where $Ax = y$, either $x \in X_0$ or $x \in X$ correspondingly and $b \in \mathcal{A}_r$, since $A(xb) = (Ax)b$ for each $x \in X_0$ or $x \in X$ respectively and $b \in \mathcal{A}_r$.

1.3. First order partial differential operators. We consider an arbitrary first order partial differential operator σ given by the formula

(1) $\sigma f = \sum_{j=0}^{2^r-1} i_j^*(\partial f/\partial z_{\xi(j)})\psi_j$,
where f is a differentiable $\mathcal{A}_r$-valued function on the domain U satisfying Conditions $1.1(D1, D2)$, $2 \le r$, $i_0, ..., i_{2^r-1}$ are the standard generators of the Cayley-Dickson algebra $\mathcal{A}_r$, ψ_j are real constants so that $\sum_j \psi_j^2 > 0$, $\xi : \{0, 1, ..., 2^r - 1\} \to \{0, 1, ..., 2^r - 1\}$ is a surjective bijective mapping, i.e. ξ belongs to the symmetric group S_{2^r} (see also §2 in [16]).

Making the substitution $f \mapsto fi_{2^r}$ and using the embedding $\mathcal{A}_r \hookrightarrow \mathcal{A}_{r+1}$ it is always possible to relate the case of the operator σ with $\psi_0 \ne 0$ and σ with $\psi_j \ne 0$ only for some $j > 0$, where i_{2^r} is the doubling generator.

For an ordered product $\{_1f..._kf\}_{q(k)}$ of differentiable functions $_sf$ we put

(2) ${}^s\sigma\{_1f..._kf\}_{q(k)} = \sum_{j=0}^n i_j^*\{_1f...(\partial_s f/\partial z_{\xi(j)})..._kf\}_{q(k)}\psi_j$,
where a vector $q(k)$ indicates on an order of the multiplication in the curled brackets (see also §2 [10, 9]), so that

(3) $\sigma\{_1f..._kf\}_{q(k)} = \sum_{s=1}^k {}^s\sigma\{_1f..._kf\}_{q(k)}$.

Symmetrically other operators

(4) $\hat{\sigma} f = \sum_{j=0}^{2^r-1}(\partial f/\partial z_{\xi(j)})i_j\psi_j$,
are defined. Therefore, these operators are related:

(5) $(\sigma f)^* = \hat{\sigma}(f^*)$.

Operators σ given by (1) are right $\mathcal{A}_r$ linear.

2. Integral operators. We consider integral operators of the form:

$$(1) \quad \mathsf{K}(x, y) = \mathsf{F}(x, y) + {}_\sigma\int_x^\infty \mathsf{F}(z, y)\mathsf{N}(x, z, y)dz,$$

where σ is an $\mathbf{R}$-linear partial differential operator as in §1.3 and ${}_\sigma\int$ is the non-commutative line integral (anti-derivative operator) over the Cayley-Dickson algebra $\mathcal{A}_r$ from §§2.5 and 2.23 [16] or 4.2.5 [11], where F and K are continuous functions with values in the Cayley-Dickson algebra $\mathcal{A}_r$ or more generally in the real algebra $Mat_s(\mathcal{A}_r)$ of $s \times s$ matrices with entries in $\mathcal{A}_r$. For definiteness we take ${}_{\partial/\partial x_0}\int g(z)dz = \int g(z)dz$ and ${}_\sigma\int g(z)dz$ calculated with the help of the left algorithm 2.6 [10, 9] or $1.2.6(LI)$ [11]. In this case ${}_\sigma\int g(z)dz$ is the $\mathcal{A}_r$ right linear operator in accordance with Lemma 1.2.2.

If $\gamma : [a, b] \to \mathcal{A}_r$ is a function, then

(*i*) $V_a^b\gamma := \sup_P |\gamma(t_{j+1}) - \gamma(t_j)|$
is called the variation of γ on the segment $[a,b] \subset \mathbf{R}$, where the supremum is taken by all finite partitions P of the segment $[a,b]$, $P = \{t_0 = a < t_1 < ... < t_n = b\}$, $n \in \mathbf{N}$. A continuous function $\gamma : [a,b] \to \mathcal{A}_r$ with the finite variation $V_a^b\gamma < \infty$ is called a rectifiable path.

Let a domain U be provided with a foliation by locally rectifiable paths $\{\gamma^\alpha : \alpha \in \Lambda\}$, where Λ is a set (see below). We take for definiteness a canonical closed domain U in $\hat{\mathcal{A}}_r$ satisfying Conditions 1.1$(D1, D2)$ so that $\infty \in U$, where $\hat{\mathcal{A}}_r = \mathcal{A}_r \cup \{\infty\}$ denotes the one-point compactification of $\mathcal{A}_r$, $2 \le r < \infty$.

A path $\gamma :< a,b >\to U$ is called locally rectifiable, if it is rectifiable on each compact segment $[c,e] \subset< a,b >$, where $< a,b >= [a,b] := \{t \in \mathbf{R} : a \le t \le b\}$ or $< a,b >= [a,b) := \{t \in \mathbf{R} : a \le t < b\}$ or $< a,b >= (a,b] := \{t \in \mathbf{R} : a < t \le b\}$ or $< a,b >= (a,b) := \{t \in \mathbf{R} : a < t < b\}$.

A domain U is called foliated by locally rectifiable paths $\{\gamma^\alpha : \alpha \in \Lambda\}$ if $\gamma :< a_\alpha, b_\alpha >\to U$ for each α and it satisfies the following three conditions:

$(F1)$ $\bigcup_{\alpha\in\Lambda} \gamma^\alpha(< a_\alpha, b_\alpha >) = U$ and

$(F2)$ $\gamma^\alpha(< a_\alpha, b_\alpha >) \cap \gamma^\beta(< a_\beta, b_\beta >) = \emptyset$ for each $\alpha \ne \beta \in \Lambda$.

Moreover, if the boundary $\partial U = cl(U) \setminus Int(U)$ of the domain U is non-void then

$(F3)$ $\partial U = (\bigcup_{\alpha\in\Lambda_1} \gamma^\alpha(a_\alpha)) \cup (\bigcup_{\beta\in\Lambda_2} \gamma^\beta(b_\beta))$,

where $\Lambda_1 = \{\alpha \in \Lambda :< a_\alpha, b_\beta >= [a_\alpha, b_\beta >\}$, $\Lambda_2 = \{\alpha \in \Lambda :< a_\alpha, b_\beta >=< a_\alpha, b_\beta]\}$. For the canonical closed subset U we have $cl(U) = U = cl(Int(U))$, where $cl(U)$ denotes the closure of U in $\mathcal{A}_v$ and $Int(U)$ denotes the interior of U in $\mathcal{A}_v$. For convenience one can choose C^1 foliation, i.e. each γ^α is of class C^1. When U is with non-void boundary we choose a foliation family such that $\bigcup_{\alpha\in\Lambda} \gamma(a_\alpha) = \partial U_1$, where a set ∂U_1 is open in the boundary ∂U and so that $w|_{\partial U_1}$ would be a sufficient initial condition to characterize a unique branch of an anti-derivative $w(x) = \mathcal{I}_\sigma f(x) = {}_\sigma\int_{{}_0x}^x f(z)dz$,
where ${}_0x \in \partial U_1$, $x \in U$, $\gamma^\alpha(t_0) = {}_0x$, $\gamma^\alpha(t) = x$ for some $\alpha \in \Lambda$, t_0 and $t \in< a_\alpha, b_\alpha >$,

${}_\sigma\int_{{}_0x}^x f(z)dz = {}_\sigma\int_{\gamma^\alpha|_{[t_0,t]}} f(z)dz.$

In accordance with Theorems 2.5 and 2.23 [16] or 4.2.5 and 4.2.23 [11] the equality

(2) $\sigma_x \ {}_\sigma\int_{{}_0x}^x g(z)dz = g(x)$

is satisfied for a continuous function g on a domain U as in §1 and a foliation as above.

Particularly in the class of $\mathcal{A}_r$ holomorphic functions in the domain satisfying Conditions 1.1$(D1, D2)$ this line integral depends only on initial and

final points due to the homotopy theorem [10, 9].

We denote by $\mathbf{P} = \mathbf{P}(U)$ the family of all locally rectifiable paths $\gamma :< a_\gamma, b_\gamma > \to U$ supplied with the family of pseudo-metrics

(3) $\rho^{a,b,c,d}(\gamma, \omega) := |\gamma(a) - \omega(c)| + \inf_\phi V_a^b(\gamma(t) - \omega(\phi(t))$

where the infimum is taken by all diffeomorphisms $\phi : [a, b] \to [c, d]$ so that $\phi(a) = c$ and $\phi(b) = d$, $a < b$, $c < d$, $[a, b] \subset < a_\gamma, b_\gamma >$, $[c, d] \subset < a_\omega, b_\omega >$. We take a foliation such that Λ is a uniform space and the limit

(4) $\lim_{\beta\to\alpha} \rho^{a,b,a,b}(\gamma^\beta, \gamma^\alpha) = 0$

is zero for each $[a, b] \subset < a_\alpha, b_\alpha >$.

For example, we can take $\Lambda = \mathbf{R}^{n-1}$ for a foliation of the entire Cayley-Dickson algebra $\mathcal{A}_r$, where $n = 2^r$, $t \in \mathbf{R}$, $\alpha \in \Lambda$, so that $\bigcup_{\alpha\in\Lambda} \gamma^\alpha([0, \infty)) = \{z \in \mathcal{A}_r : \ Re((z - y)v^*) \ge 0\}$ and $\bigcup_{\alpha\in\Lambda} \gamma^\alpha((-\infty, 0]) = \{z \in \mathcal{A}_r : \ Re((z - y)v^*) \le 0\}$ are two real half-spaces, where $v, y \in \mathcal{A}_r$ are marked Cayley-Dickson numbers and $v \ne 0$. Particularly, we can choose the foliation such that $\gamma^\alpha(0) = y + \alpha_1 v_1 + ... + \alpha_{2^r-1} v_{2^r-1}$ and $\gamma^\alpha(t) = tv_0 + \gamma^\alpha(0)$ for each $t \in \mathbf{R}$, where $v_0, ..., v_{2^r-1}$ are $\mathbf{R}$-linearly independent vectors in $\mathcal{A}_r$.

Therefore, the expression

$$(5) \quad {}_\sigma\int_x^\infty g(z)dz := {}_\sigma\int_{\gamma^\alpha|_{[t_x, b_\alpha>}} g(z)dz$$

denotes a non-commutative line integral over $\mathcal{A}_r$ along a path γ^α so that $\gamma^\alpha(t_x) = x$ and $\lim_{b\to b_\alpha} \gamma^\alpha(t) = \infty$ for an integrable function g, where $t_x \in < a_\alpha, b_\alpha >$, $\alpha \in \Lambda$, $a_\alpha = a_{\gamma^\alpha}$, $b_\alpha = b_{\gamma^\alpha}$. It is sometimes convenient to use the line integral

$$(6) \quad {}_\sigma\int_x^{-\infty} g(z)dz := {}_\sigma\int_{\gamma^\alpha|_{<a_\alpha, t_x]}} g(z)dz,$$

when $\lim_{a\to a_\alpha} \gamma^\alpha(t) = \infty$.

If ${}_\sigma\int g(z)dz = \sigma^{-1}g$ corresponds to the right (or left) algorithm of the non-commutative line integration over the Cayley-Dickson algebra $\mathcal{A}_r$, then $i_{2^r}[({}_\sigma\int g(z)dz)i_{2^r}^*]$ corresponds to the left (or right correspondingly) algorithm such that

$i_{2^r}[(\sigma f)i_{2^r}^*] = \sum_{j=0}^{2^r-1} (\partial f^*/\partial z_{\xi(j)})i_j\psi_j = \hat{\sigma}(f^*)$,

where f is a differentiable $\mathcal{A}_r$ valued function.

For example, one can take a function $g(z)$ as $\mathsf{F}(z, y)\mathsf{K}(x, z)$ depending on additional parameters $x, y, t, ...$.

A function F has the decomposition

(7) $\mathsf{F} = \sum_{j=0}^{2^r-1} \mathsf{F}_j i_j$,

where each function F_j is real-valued for a function F.

Put for convenience $\sigma^0 = I$, where I denotes the unit operator, σ^m denotes the m-th power of σ for each non-negative integer $0 \le m \in \mathbf{Z}$.

3. Proposition. *A family $\mathcal{D}_r$ of all differential operators with constant $\mathcal{A}_r$ coefficients is a power associative real algebra with a center $Z(\mathcal{D}_r)$ consisting of all differential operators with real coefficients and with a unit element I.*

Proof. Let A be a differential operator with constant $\mathcal{A}_r$ coefficients, then it can be written in the form:

(1) $Af = \sum_{j,s=0}^{2^r-1}[A_{j,s}\pi_s f]i_j^*$,

where $A_{j,s}$ is a differential operator with real coefficients for each j:

(2) $A_{j,s} = \sum_\alpha a_{j,s,\alpha}\partial^\alpha$,

$a_{j,s,\alpha} \in \mathbf{R}$ for each $\alpha = (\alpha_0, ..., \alpha_{2^r-1})$, $\partial^\alpha = \frac{\partial^{|\alpha|}}{\partial x_0^{\alpha_0}...\partial x_{2^r-1}^{\alpha_{2^r-1}}}$, $|\alpha| = \alpha_0 + ... + \alpha_{2^r-1}$, $\alpha_k = 0, 1, 2, ...$ is a non-negative integer for each $k = 0, ..., 2^r - 1$, $\pi_s : X \to X_s$ denotes the $\mathbf{R}$-linear projection operator (see §1.2),

(3) $Af = \sum_{j,s=0}^{2^r-1}(A_{j,s}f_s)i_j^*$

for each f in a domain $Dom(A)$ of the operator A, where

(4) $f = \sum_{s=0}^{2^r-1} f_s i_s$,

f_s is a real-valued function for each s, $f_s = \pi_s f$. If $P_n(z)$ is a polynomial with $\mathcal{A}_r$ coefficients of the variable $z \in \mathcal{A}_r$ (see also §2.1 [9, 10]), then

(5) $AP_n(z) = \sum_{j,s=0}^{2^r-1}\sum_{\alpha,|\alpha|\le n} a_{j,s,\alpha}[\partial^\alpha(P_n)_s(z)]i_j^*$,

consequently, P_n belongs to the domain $Dom(A)$ of the operator A. Thus $Dom(A)$ is non-void for each differential operator $A \in \mathcal{D}_r$. From Formulas $(1-4)$ we get, that the sum $A+B$ and the product BA of any differential operators $A, B \in \mathcal{D}_r$ belongs to $\mathcal{D}_r$, since

(6) $BAf = B(Af) = \sum_{k,j,s=0}^{2^r-1}(-1)^{sign(j)}(B_{k,j}A_{j,s}f_s)i_k^* = Cf$ with

(7) $C_{k,s} = \sum_{j=0}^{2^r-1}(-1)^{sign(j)}B_{k,j}A_{j,s}$,

where $sign(x) = 0$ for $x = 0$, $sign(x) = 1$ for $x > 0$, $sign(x) = -1$ for $x < 0$. Certainly $tA \in \mathcal{D}_r$ for each $t \in \mathbf{R}$ and $A \in \mathcal{D}_r$. From Formulas $4(7-9)$ [16] or $4.2.4(7-9)$ [11] we deduce that

(8) $A^2 f = A(Af) = \sum_{j=0}^{2^r-1}(A_j^2 f)i_j^2 + \sum_{j=0}^{2^r-1}[(A_0A_j + A_jA_0)f]i_j^*$

for each $f \in Dom(A^2)$, since $Ix = \sum_{j=0}^{2^r-1}\pi_j(x)i_j$ for each $x \in X$, where

(9) $A_j f = \sum_{s=0}^{2^r-1} A_{j,s}f_s$,

since $(i_j^*)^2 = i_j^2$. Therefore, by induction we infer that $A^{2k} = (A^2)^k$ and $A^{2k+1} = (A^2)^kA = A(A^2)^k$ for each natural number k, where $A^0 = I$ denotes the unit operator, i.e. $If = f$ for each function $f : \mathcal{A}_r \to \mathcal{A}_r$. Thus the algebra $\mathcal{D}_r$ is power associative:

(10) $A^kA^m = A^{k+m}$ for each natural numbers $k, m \in \mathbf{N}$ and every $A \in \mathcal{D}_r$. Moreover it contains the unit element I.

If $A \in Z(\mathcal{D}_r)$, i.e. an operator A commutes with every differential operator $B \in \mathcal{D}_r$, $[A, B] := AB - BA = 0$, then it commutes with all generators $\{i_j, \partial_{x_k}i_j^* : j, k = 0, ..., 2^r - 1\}$ of the algebra $\mathcal{D}_r$. From Formulas $(1-4, 6, 7)$

it follows that $a_{j,s,\alpha} = 0$ for $j > 0$, where $a_{j,s,\alpha} \in \mathbf{R}$ for each j, s and α. That is, all the coefficients $a_{j,s,\alpha} i_j^*$ of the operator A are real, since $i_0 = 1$.

3.1. Corollary. *Let $A \in \mathcal{D}_r$ be a differentiable operator with constant $\mathcal{A}_r$ coefficients and $\pi_0 : X \to X_0$ be the $\mathbf{R}$ linear projection (see §1.2), then $[(I - \pi_0)A]^{2k}$ is with real coefficients for each natural number $k \in \mathbf{N}$.*

Proof. In view of Formulas 3(1, 9) the equality is valid:

(1) $(I - \pi_0)Af = \sum_{j=1}^{2^r-1} (A_j f) i_j^*$, consequently,

(2) $[(I - \pi_0)A]^2 f = \sum_{j=1}^{2^r-1} (A_j^2 f) i_j^2$.

That is, $[(I - \pi_0)A]^2$ is the differential operator with real coefficients, since $i_j^2 \in \mathbf{R}$ for each j.

3.2. Remark. Mention that in the considered case $\psi_0 = 0$ and the operator σ^2 is with real coefficients due to Corollary 3.1, hence the square of the anti-derivative operator $({}_\sigma \int^x)^2$ or σ_x^{-2} is also with real coefficients, when each ψ_j is a real constant and $\psi_0 = 0$.

4. Proposition. *Let*

$$(1) \quad \lim_{z\to\infty} {}^1\sigma_z^k \, {}^2\sigma_x^s \, {}^2\sigma_z^n \mathsf{F}(z, y)\mathsf{K}(x, z) = 0$$

for each x, y in a domain U satisfying Conditions 1.1$(D1, D2)$ with $\infty \in U$ and every non-negative integers $0 \le k, s, n \in \mathbf{Z}$ such that $k+s+n \le m$. Suppose also that ${}_\sigma \int_x^\infty \partial_x^\alpha \partial_y^\beta \partial_z^\omega [\mathsf{F}(z, y)\mathsf{K}(x, z)]dz$ converges uniformly by parameters x, y on each compact subset $W \subset U \subset \mathcal{A}_r^2$ for each $|\alpha| + |\beta| + |\omega| \le m$, where $\alpha = (\alpha_0, ..., \alpha_{2^r-1})$, $|\alpha| = \alpha_0 + ... + \alpha_{2^r-1}$, $\partial_x^\alpha = \partial^{|\alpha|}/\partial x_0^{\alpha_0} ... \partial x_{2^r-1}^{\alpha_{2^r-1}}$. Then the non-commutative line integral ${}_\sigma \int_x^\infty \mathsf{F}(z, y)\mathsf{K}(x, z)dz$ from §2 satisfies the identities:

$$(2) \quad \sigma_x^m \, {}_\sigma \int_x^\infty \mathsf{F}(z, y)\mathsf{K}(x, z)dz = {}^2\sigma_x^m \, {}_\sigma \int_x^\infty \mathsf{F}(z, y)\mathsf{K}(x, z)dz + A_m(\mathsf{F}, \mathsf{K})(x, y),$$

$$(3) \quad {}^1\sigma_z^m \, {}_\sigma \int_x^\infty \mathsf{F}(z, y)\mathsf{K}(x, z)dz = (-1)^m \, {}^2\sigma_z^m \, {}_\sigma \int_x^\infty \mathsf{F}(z, y)\mathsf{K}(x, z)dz + B_m(\mathsf{F}, \mathsf{K})(x, y),$$

where

$$(4) \quad A_m(\mathsf{F}, \mathsf{K})(x, y) = - \, {}^2\sigma_x^{m-1} [\mathsf{F}(x, y)\mathsf{K}(x, z)]|_{z=x} + \sigma_x \; A_{m-1}(\mathsf{F}, \mathsf{K})(x, y)$$

for $m \ge 2$,

$$(5) \quad B_m(\mathsf{F}, \mathsf{K})(x, y) = (-1)^m \, {}^2\sigma_z^{m-1} \mathsf{F}(x, y)\mathsf{K}(x, z)|_{z=x} + [{}^1\sigma_z \; B_{m-1}(\mathsf{F}(z, y), \mathsf{K}(x, z))]|_{z=x}$$

for $m \ge 2$,

(6) $A_1(\mathsf{F}, \mathsf{K})(x, y) = B_1(\mathsf{F}, \mathsf{K})(x, y) = -\mathsf{F}(x, y)\mathsf{K}(x, x)$,

σ_x is an operator σ acting by the variable $x \in U \subset \mathcal{A}_r$.

Proof. From the conditions of this proposition and the theorem about differentiability of improper integrals by parameters we have the equality

$_\sigma\int_x^\infty \partial_x^\alpha\partial_y^\beta\partial_z^\omega[\mathsf{F}(z,y)\mathsf{K}(x,z)]dz = \partial_x^\alpha\partial_y^\beta\ {_\sigma\int_x^\infty} \partial_z^\omega[\mathsf{F}(z,y)\mathsf{K}(x,z)]dz$
for each $|\alpha|+|\beta|+|\omega|\le m$ (see also Part IV, Chapter 2, §4 in [6]).

In view of Theorems 2.5 and 2.23 and Corollary 2.6 [16] or 4.2.5 and 4.2.23 and Corollary 4.2.6 [11] the equalities

$$(7)\quad \sigma_x\ {_\sigma\int_x^\infty} g(z)dz = -g(x) \text{ and}$$

$$(8)\quad {_\sigma\int_{{_0}x}^x}[\sigma_z f(z)]dz = f(x) - f({_0}x)$$

are satisfied for each continuous function g and continuously differentiable functions g and f, where ${_0}x$ is a marked point in U,

$$(9)\quad {^1\sigma_z}\ {_\sigma\int_x^\infty} \mathsf{F}(z,y)\mathsf{K}(x,z)dz := \sum_{j=0}^{2^r-1}\ {_\sigma\int_x^\infty}\{i_j^*[(\partial\mathsf{F}(z,y)/\partial z_{\xi(j)})\mathsf{K}(x,z)]\psi_j\}dz \text{ and}$$

$$(10)\quad {^2\sigma_z}\ {_\sigma\int_{{_0}x}^x} \mathsf{F}(z,y)\mathsf{K}(x,z)dz := \sum_{j=0}^{2^r-1}\ {_\sigma\int_{{_0}x}^x}\{i_j^*[\mathsf{F}(z,y)(\partial\mathsf{K}(x,z)/\partial z_{\xi(j)})]\psi_j\}dz \text{ and}$$

$$(11)\quad {^2\sigma_x}\ {_\sigma\int_x^\infty} \mathsf{F}(z,y)\mathsf{K}(x,z)dz := \sum_{j=0}^{2^r-1}\ {_\sigma\int_x^\infty}\{i_j^*[\mathsf{F}(z,y)(\partial\mathsf{K}(x,z)/\partial x_{\xi(j)})]\psi_j\}dz.$$

Therefore, from Equalities $(7,8)$, 1.3(3) and 2(5) and Condition (1) one gets:

$$(12)\quad \sigma_x\ {_\sigma\int_x^\infty} \mathsf{F}(z,y)\mathsf{K}(x,z)dz = {^2\sigma_x}\ {_\sigma\int_x^\infty} \mathsf{F}(z,y)\mathsf{K}(x,z)dz - \mathsf{F}(x,y)\mathsf{K}(x,x),$$

since $\mathsf{F}(z,y)\mathsf{K}(x,z)|_x^\infty = -\mathsf{F}(x,y)\mathsf{K}(x,x)$, that demonstrates Formula (2) for $m=1$ and $A_1 = -\mathsf{F}(x,y)\mathsf{K}(x,x)$. The induction by $p=2,...,m$ gives:

$$(13)\quad \sigma_x^p\ {_\sigma\int_x^\infty} \mathsf{F}(z,y)\mathsf{K}(x,z)dz =$$

$$\sigma_x\ [{^2\sigma_x^{p-1}}\ {_\sigma\int_x^\infty} \mathsf{F}(z,y)\mathsf{K}(x,z)dz] + \sigma_x\ A_{p-1}(\mathsf{F},\mathsf{K})(x,y)$$

$$= {^2\sigma_x^p}\ {_\sigma\int_x^\infty} \mathsf{F}(z,y)\mathsf{K}(x,z)dz$$

$$+\ {^2\sigma_x^{p-1}}A_1(\mathsf{F}(x,y),\mathsf{K}(x,z)|_{z=x}) + \sigma_x\ A_{p-1}(\mathsf{F},\mathsf{K})(x,y).$$

Substituting in these expressions A_1 one gets Formulas $(2,4)$ for each $m\ge 2$. Then from Formulas $(7,8)$ and Condition (1) we infer also that

$$(14)\quad {^1\sigma_z}\ {_\sigma\int_x^\infty} \mathsf{F}(z,y)\mathsf{K}(x,z)dz = -\ {^2\sigma_z}\ {_\sigma\int_x^\infty} \mathsf{F}(z,y)\mathsf{K}(x,z)dz + \mathsf{F}(z,y)\mathsf{K}(x,z)|_x^\infty$$

$$= -\mathsf{F}(x,y)\mathsf{K}(x,x) -\ {^2\sigma_z}\ {_\sigma\int_x^\infty} \mathsf{F}(z,y)\mathsf{K}(x,z)dz,$$

that gives Formulas (3) for $m=1$ and (6). Then we deduce Formulas $(3,5)$ by induction on $p=2,...,m$:

$$(15)\quad {^1\sigma_z^p}\ {_\sigma\int_x^\infty} \mathsf{F}(z,y)\mathsf{K}(x,z)dz =$$

$$(-1)^{p-1}\ {}^1\sigma_z\ [{}^2\sigma_z^{p-1}\ {}_\sigma\int_x^\infty \mathsf{F}(z,y)\mathsf{K}(x,z)dz] + {}^1\sigma_z\ B_{p-1}(\mathsf{F}(z,y),\mathsf{K}(x,z))|_{z=x}$$
$$= (-1)^p\ {}^2\sigma_z^p\ {}_\sigma\int_x^\infty \mathsf{F}(z,y)\mathsf{K}(x,z)dz$$
$$+(-1)^p\ {}^2\sigma_z^{p-1}\mathsf{F}(z,y)\mathsf{K}(x,z)|_{z=x} + {}^1\sigma_z\ B_{p-1}(\mathsf{F}(z,y),\mathsf{K}(x,z))|_{z=x}.$$

4.1. Remark. The center of the Cayley-Dickson algebra $\mathcal{A}_r$ for $r\ge 2$ is the real field $\mathbf{R}$.

Let $Mat_s(\mathcal{A}_r)$ denote the left and right $\mathcal{A}_r$ module of square $s\times s$ matrices with entries in $\mathcal{A}_r$. It is possible to consider the quaternion skew field $\mathbf{H}_{J,K,L}$ with generators J,K,L realized as 4×4 square real matrices. Putting $i_jY = (y_{k,l}i_j) = Yi_j$ for each real matrix Y with elements $y_{k,l}\in\mathbf{R}$ for each k,l and every generator i_j of the Cayley-Dickson algebra we naturally obtain the quaternionified algebra $(\mathcal{A}_r)_{\mathbf{H}_{J,K,L}}$ (see also [7, 8]). On the other hand, $\mathbf{H}\subset Mat_4(\mathbf{R})$, consequently, $Mat_s((\mathcal{A}_r)_{\mathbf{H}_{J,K,L}})\subset Mat_{4s}(\mathcal{A}_r)$.

5. Corollary. *If suppositions of Proposition 4 are satisfied, then*

(1) $A_2(\mathsf{F},\mathsf{K})(x,y) = -\sigma_x[\mathsf{F}(x,y)\mathsf{K}(x,x)] -\ {}^2\sigma_x[\mathsf{F}(x,y)\mathsf{K}(x,z)]|_{z=x}$,

(2) $A_3(\mathsf{F},\mathsf{K})(x,y) = -\sigma_x^2[\mathsf{F}(x,y)\mathsf{K}(x,x)]$
$-\sigma_x(\ {}^2\sigma_x[\mathsf{F}(x,y)\mathsf{K}(x,z)]|_{z=x}) -\ {}^2\sigma_x^2[\mathsf{F}(x,y)\mathsf{K}(x,z)]|_{z=x}$,

(3) $B_2(\mathsf{F},\mathsf{K})(x,y) = -{}^1\sigma_x[\mathsf{F}(x,y)\mathsf{K}(x,x)] +\ {}^2\sigma_z[\mathsf{F}(x,y)\mathsf{K}(x,z)]|_{z=x}$,

(4) $B_3(\mathsf{F},\mathsf{K})(x,y) = -{}^1\sigma_x^2[\mathsf{F}(x,y)\mathsf{K}(x,x)]$
$+{}^1\sigma_x({}^2\sigma_z[\mathsf{F}(x,y)\mathsf{K}(x,z)]|_{z=x}]) -\ {}^2\sigma_z^2[\mathsf{F}(x,y)\mathsf{K}(x,z)]|_{z=x}$.

(5) $A_2(\mathsf{F},\mathsf{K})(x,y) - B_2(\mathsf{F},\mathsf{K})(x,y) = -2\ {}^2\sigma_x[\mathsf{F}(x,y)\mathsf{K}(x,x)]$,
where $\sigma_x\mathsf{K}(x,x) = [\sigma_x\mathsf{K}(x,z)+\sigma_z\mathsf{K}(x,z)]|_{z=x}$,

(6) $A_3(\mathsf{F},\mathsf{K})(x,y) - B_3(\mathsf{F},\mathsf{K})(x,y) = -(3\ {}^2\sigma_x^2 + {}^2\sigma_x\ {}^2\sigma_z + 2\ {}^2\sigma_z\ {}^2\sigma_x)[\mathsf{F}(x,y)\mathsf{K}(x,z)]|_{z=x}$
$-(2\ {}^1\sigma_x\ {}^2\sigma_x +\ {}^2\sigma_x\ {}^1\sigma_x)[\mathsf{F}(x,y)\mathsf{K}(x,x)]$.

Particularly, if either p is even and $\psi_0 = 0$, or $\mathsf{F}\in Mat_s(\mathbf{R})$ and $\mathsf{K}\in Mat_s(\mathcal{A}_r)$, then

(7) ${}^2\sigma_x^p[\mathsf{F}(z,y)\mathsf{K}(x,z)] = \mathsf{F}(z,y)\sigma_x^p\mathsf{K}(x,z)$ *and* ${}^2\sigma_z^p[\mathsf{F}(z,y)\mathsf{K}(x,z)] = \mathsf{F}(z,y)\sigma_z^p\mathsf{K}(x,z)$.

Proof. Formulas $(1-6)$ follow from Equalities $4(4-6)$. In particular, if p is even and $\psi_0 = 0$, $p = 2k$, $k\in\mathbf{N}$, then in accordance with Corollary 3.1 and Formulas $4(7-9)$ [16] or $4.2.4(7-9)$ [11]

$\sigma_x^p f(x) = A^k f(x)$

for p times differentiable function $f: U\to\mathcal{A}_r$, where

$Af = \sum_j b_j\partial^2 f(x)/\partial x_j^2$, $b_j = i_{\xi^{-1}(j)}^2\in\mathbf{R}$.

Evidently the operators ${}^2\sigma_x^p$ and ${}^2\sigma_z^p$ commute with the left multiplication on $\mathsf{F}(z,y)\in Mat_s(\mathbf{R})$ in accordance with §4.1, that is ${}^2\sigma_x^p[\mathsf{F}(z,y)\mathsf{K}(x,z)] = \mathsf{F}(z,y)\sigma_x^p\mathsf{K}(x,z)$ and ${}^2\sigma_z^p[\mathsf{F}(z,y)\mathsf{K}(x,z)] = \mathsf{F}(z,y)\sigma_z^p\mathsf{K}(x,z)$.

6. Remark. Expressions of functions A_m and B_m depend not only on F and K, but also on σ, that is on coefficients ψ_j for $j = 0,...,2^r-1$ with $2\le r$ in accordance with §§4 and 5.

7. Method of non-commutative integration of vector partial differential equations.

Let an equation over the Cayley-Dickson algebra $\mathcal{A}_r$ be given in the non-commutative line integral form:

$$(1) \quad \mathsf{K}(x,y) = \mathsf{F}(x,y) + \mathsf{p} \ {}_{\sigma}\int_x^{\infty} \mathsf{F}(z,y)N(x,z,y)dz,$$

where K, F and N are continuous integrable functions as in §2, $\mathsf{p} \in \mathbf{R} \setminus \{0\}$ is a non-zero real constant. These functions may depend on additional parameters $t, \tau, \dots$.

The first step consists in a concretization of a function N and its expression throughout F and K. Suppose that an operator

(2) $(I - \mathsf{A}_x)\mathsf{F}(x,y) = \mathsf{K}(x,y)$ is invertible, so that $(I - \mathsf{A}_x)^{-1}$ is continuous, where I denotes the unit operator, while

$\mathsf{A}_x = -\mathsf{p} \ {}_{\sigma}\int_x^{\infty} \mathsf{F}(z,y)\mathsf{N}(x,z,y)dz$

is an operator acting by variables x.

On the second step two $\mathbf{R}$-linear differential or partial differential operators L_s over the Cayley-Dickson algebra $\mathcal{A}_r$ are given:

(3) $L_s f = \sum_j i_j^*(L_{s,j} f)$,

where f is a differentiable function in the domain of L_s, $L_{s,j}g$ is real-valued function for each $ord(L_s)$ times differentiable real-valued function g in the domain of $L_{s,j}$ for every j (see also Formulas 3(1 − 4)), where $ord(L_s)$ denotes the order of the partial differential operator L_s. On a function F the conditions either:

(4) $L_s\mathsf{F} = 0$

for each $s = 1, 2$, or

(5) $\sum_{j \in \Psi_k} i_j^*[\ {}_jc(L_{s,0}\mathsf{F}) + L_{s,j}\mathsf{F}] = 0$

for each $s = 1, 2$ and $1 \le k \le m$ are imposed, where ${}_jc$ are constants ${}_jc \in \mathcal{A}_r$, $\Psi_k \subset \{0, 1, ..., 2^r - 1\}$ for each k, $\bigcup_k \Psi_k = \{0, 1, ..., 2^r - 1\}$, $\Psi_k \cap \Psi_l = \emptyset$ for each $k \ne l$, $1 \le m \le 2^r$. Coefficients ${}_jc$ or operators $L_{s,j}$ may be zero for some j.

On the third step a function K is calculated from Equation (2).

Acting by the operator L_s from the left on (2) and using Conditions either (4) or (5) one gets either

(6) $L_s[(I - A_x)\mathsf{K}] = 0$ or the equalities:

(7) $\sum_{j \in \Psi_k} i_j^*\{\ {}_jcL_{s,0}[(I - \mathsf{A}_x)\mathsf{K}] + L_{s,j}[(I - \mathsf{A}_x)\mathsf{K}]\} = 0$ for each $k = 1, ..., m$.

Therefore, using Conditions either (4) or (5) we infer that

(8) $(I - \mathsf{A}_x)(L_s\mathsf{K}) = R_s(\mathsf{K})$ for $s = 1, 2$,

where an operator

(9) $R_s(f) = (I - \mathsf{A}_x)(L_s f) - L_s[(I - \mathsf{A}_x)f]$
is formed with the help of commutators $[A,B] = AB - BA$ and anti-commutators $\{A,B\} = AB + BA$ of operators $(I - (\mathsf{A}_x)_0)$, $(\mathsf{A}_x)_k$, $L_{s,j}$, $k,j = 0,...,2^r - 1$. A function N and operators L_1 and L_2 are chosen such that

(10) $R_s(\mathsf{K}) = (I - \mathsf{A}_x)M_s(\mathsf{K})$ for $s = 1,2$,
where $M_s(\mathsf{K})$ are functionals of K which generally may be non-$\mathbf{R}$-linear. In view of Condition (2) the function K must satisfy partial differential equations

(11) $L_s\mathsf{K} - M_s(\mathsf{K}) = 0$ for $s = 1,2$,
which generally may be non-$\mathbf{R}$-linear. Thus each solution K of the $\mathbf{R}$-linear integral equation (1) is also the solution of partial differential equations (11). Frequently particular cases are considered, when Equations (4) correspond to an eigenvalue problem for $s = 1$ and to an evolution in time problem for $s = 2$. Generally operators $L_{s,0},...,L_{s,2^r-1}$ can be chosen $\mathcal{A}_r$ vector independent and ${}_jc \in \mathbf{R}i_{k(j)}$ for each j and $s = 1,2$ with $k = k(j) \in \{0,1,..,2^r - 1\}$.

The Euclidean space $\mathbf{R}^{2^r}$ is the real shadow of the Cayley-Dickson algebra $\mathcal{A}_r$, that is, by the definition $\mathcal{A}_r$ considered as the $\mathbf{R}$ linear space is isomorphic with $\mathbf{R}^{2^r}$. The Lebesgue (non-negative) measure μ on the Borel σ-algebra $\mathcal{B}(\mathbf{R}^{2^r})$ of the Euclidean space $\mathbf{R}^{2^r}$ induces the Lebesgue measure on $\mathcal{B}(\mathcal{A}_r)$. Therefore, the Hilbert space $X = L^2(\mathcal{A}_r,\mu,\mathcal{A}_r)$ of all Lebesgue measurable functions $f : \mathcal{A}_r \to \mathcal{A}_r$ with integrable square module $|f|^2$, i.e.

(HN) $\quad \|f\|^2 := \int_{\mathcal{A}_r} |f(z)|^2\mu(dz) < \infty$, and with the $\mathcal{A}_r$ valued scalar product

(SP) $\quad (f,g) := \int_{\mathcal{A}_r} f^*(z)g(z)\mu(dz)$
exists. Analogously the Hilbert space $X = L^2(\mathcal{A}_r,\mu,Mat_s(\mathcal{A}_r))$ is defined with $\sum_{j,k=0}^s f^*_{j,k}(z)g_{j,k}(z)$ instead of $f^*(z)g(z)$ in the integral in Formula (SP), where $f_{j,k} \in \mathcal{A}_r$ denotes a matrix element at the intersection of row j with column k of a matrix $f \in Mat_s(\mathcal{A}_r)$.

Let A be an $\mathbf{R}$ linear $\mathcal{A}_r$ additive operator $A : \mathsf{D}(X) \to Y$, where $\mathsf{D}(A)$ is a domain of A dense in X, $\mathsf{D}(A) \subset X$, X and Y are Hilbert spaces over $\mathcal{A}_r$. Then its adjoint operator A^* is defined on a domain consisting of all those vectors $y \in Y$ such that for some vector $z \in X$ the equality $(x,z) = (Ax,y)$ is valid for all $x \in \mathsf{D}(A)$. For such $y \in Y$ we put $A^*y = z$. If $A^* = A$ we say that A is self-adjoint. Thus $(Ax,y) = (x,A^*y)$ for all $x \in \mathsf{D}(A)$ and $y \in \mathsf{D}(A^*)$.

If an operator A is self-adjoint $A = A^*$, then $(Ax,y) = (x,Ay)$ for all $x,y \in \mathsf{D}(A)$.

When $\mathsf{D}(A)$ is dense in X and $(Ax,y) = (x,Ay)$ for all $x,y \in \mathsf{D}(A)$, we say that A is symmetric. A self-adjoint operator is maximal symmetric.

We define a self-adjoint operator A to be positive, when $(Ax,x) \geq 0$ for

each $x \in \mathsf{D}(A)$.

An $\mathbf{R}$ linear $\mathcal{A}_r$ additive operator A is called invertible if it is densely defined and one-to-one and has dense range $\mathcal{R}(A)$.

The operator A^*A is self-adjoint and positive. If A is invertible, then $(A^*A)^{-1}A^* \subseteq A^{-1}$ (see also [3, 15]).

If an expression of the form

(12) $\sum_k [(I - A_x)\ _kf(x,y)]\ _kg(y) = u(x,y)$

will appear on a domain U, which need to be inverted we consider the case when

(RS) $(I - A_x)$ is either right strongly $\mathcal{A}_r$ linear, or right $\mathcal{A}_r$ linear and $_kf \in X_0$ for each k, or $\mathbf{R}$ linear and $_kg(y) \in \mathbf{R}$ for each k and every $y \in U$, at each point $x \in U$.

When an operator $(I - A_x)$ is invertible and Condition (RS) is satisfied, Equation (12) can be resolved:

(13) $\sum_k\ _kf(x,y)\ _kg(y) = (I - A_x)^{-1}u(x,y)$.

If condition (RS) is not fulfilled, the corresponding system of equations in real components $(A_x)_{j,s}$, $_kf_s$ and $_kg_s$ can be considered.

If $A : X \to X$ is a bounded $\mathbf{R}$ linear operator on a Banach space X with the norm $\|A\| < 1$, then

$(I - A)^{-1} = \sum_{n=0}^{\infty} A^n$.

The anti-derivative operator $g \mapsto\ _\sigma\int_{_0x}^x g(z)dz$ is compact from $C^0(V,\mathcal{A}_r)$ into $C^0(V,\mathcal{A}_r)$ for a compact domain V in $\mathcal{A}_r$, where $C^0(V,\mathcal{A}_r)$ is the Banach space over $\mathcal{A}_r$ of all continuous functions $g : V \to \mathcal{A}_r$ supplied with the supremum norm $\|g\| := \sup_{x\in V}|g(x)|$, $_0x$ is a marked point in V, $x \in V$. Therefore, an operator A_x will be invertible with a suitable choice of a function F satisfying the system of $\mathbf{R}$ linear partial differential equations (4) or (5) and a real non-zero parameter $\mathsf{p} \neq 0$.

7.1. Proposition. *Let V be a compact domain in the Cayley-Dickson algebra $\mathcal{A}_r$ with $2 \le r$ and let its foliation be with rectifiable paths and satisfy Condition 2(4) and Λ be a compact subset in $\mathbf{R}^{2^r}$. Then the anti-derivative operator $_\sigma\int^x$ from §2 is compact from $C^0(V,\mathcal{A}_r)$ into $C^0(V,\mathcal{A}_r)$.*

Proof. As usually $C^1(V,\mathcal{A}_r)$ denotes the space of all continuously differentiable functions $f : V \to \mathcal{A}_r$ with the supremum norm

$\|f\|_{C^1} := \sup_{x\in V}|f(x)| + \sum_{j=0}^{2^r-1}\sup_{x\in V}|\partial f(x)/\partial x_j|$.

The decomposition is valid: $C^s(V,\mathcal{A}_r) = C^s(V,\mathbf{R})i_0 \oplus ... \oplus C^s(V,\mathbf{R})i_{2^r-1}$ for $s = 0, 1, ...$ On the other hand, the partial differential operators $C^1(V,\mathbf{R}) \ni f \mapsto i^*_{\xi(j)}(\partial f(x)/\partial x_j) \in C^0(V,\mathbf{R})$ are $\mathbf{R}$ linearly independent, since $Re(i_j i_k^*) = 0$ for each $j \neq k$. In view of Theorems 2.6 [9, 10] and 1.2.7, 4.2.5 and 4.2.23 and Corollary 4.2.6 [11] the anti-derivative mapping $C^0(V,\mathcal{A}_r) \ni g \mapsto$

${}_\sigma\int^x g(z)dz \in C^1(V,\mathcal{A}_r)$ is continuous, since $\sigma_x\ {}_\sigma\int^x g(z)dz = g(x)$. But the embedding $C^1(V,\mathcal{A}_r)\hookrightarrow C^0(V,\mathcal{A}_r)$ is the $\mathbf{R}$ linear $\mathcal{A}_r$ additive compact operator. Therefore, the anti-derivative operator ${}_\sigma\int^x : C^0(V,\mathcal{A}_r)\to C^0(V,\mathcal{A}_r)$ is compact.

8. Example. Let

(1) $N(x,z,y) = \mathsf{K}(x,z)$ in 7(1).

Acting from the left on both sides of Equation 7(1) one gets:

(2) $L_s\mathsf{K}(x,y) = L_s\mathsf{p}\ {}_\sigma\int_x^\infty \mathsf{F}(z,y)\mathsf{K}(x,z)dz$.

Take the hyperbolic partial differential operator

(3) $L_1 = {}_1\sigma_x^2 - {}_2\sigma_y^2$

of the second order, where the operator ${}_k\sigma$ is with coefficients ${}_k\psi_j\in\mathbf{R}$ for each $j=0,...,2^r-1$ and a transposition $\xi_k\in S_{2^r}$ (see §1.4). Condition 7(4) for $s=1$ means that $L_1F=0$, i.e.

(4) ${}_1\sigma_z^2F(z,y) = {}_2\sigma_y^2F(z,y)$.

Then due to Proposition 4 and Corollary 5 we deduce that

$({}_1\sigma_x^2 - {}_2\sigma_y^2)\mathsf{K}(x,y) = \mathsf{p}({}_1\sigma_x^2 - {}_2\sigma_y^2)\ {}_1\sigma\int_x^\infty \mathsf{F}(z,y)\mathsf{K}(x,z)dz$

$= \mathsf{p}\ {}_1^2\sigma_x^2\ {}_1\sigma\int_x^\infty \mathsf{F}(z,y)\mathsf{K}(x,z)dz + \mathsf{p}\ {}_1A_2(\mathsf{F},\mathsf{K})(x,y) - \mathsf{p}\ {}_1^1\sigma_z^2\ {}_1\sigma\int_x^\infty \mathsf{F}(z,y)\mathsf{K}(x,z)dz$

$= \mathsf{p}({}_1^2\sigma_x^2 - {}_1^2\sigma_z^2)\ {}_1\sigma\int_x^\infty \mathsf{F}(z,y)\mathsf{K}(x,z)dz + \mathsf{p}\ {}_1A_2(\mathsf{F},\mathsf{K})(x,y) - \mathsf{p}\ {}_1B_2(\mathsf{F},\mathsf{K})(x,y)$,

consequently,

(5) $(I-\mathsf{A}_x)[({}_1\sigma_x^2 - {}_2\sigma_y^2)\mathsf{K}(x,y)] = \mathsf{p}\ {}_1A_2(\mathsf{F},\mathsf{K})(x,y) - \mathsf{p}\ {}_1B_2(\mathsf{F},\mathsf{K})(x,y)$

$= -2\mathsf{p}\ {}_1^2\sigma_x[\mathsf{F}(x,y)\mathsf{K}(x,x)] = -2\mathsf{p}\ {}_1^2\sigma_x\{[(I-\mathsf{A}_x)\mathsf{K}(x,y)]\mathsf{K}(x,x)\}$,

where the terms ${}_kA_s$ and ${}_kB_s$ correspond to ${}_k\sigma$.

If the Cayley-Dickson algebra $\mathcal{A}_r$ is either with $r=2$ and $\mathsf{F}\in Mat_s(\mathbf{R})$ and $\mathsf{K}\in Mat_s(\mathcal{A}_2)$, $s\in\mathbf{N}$, or $\mathsf{F}\in\mathbf{R}$ and $\mathsf{K}\in\mathcal{A}_3$ with $r=3$, then

${}_1^2\sigma_x\{[(I-\mathsf{A}_x)\mathsf{K}(x,y)]\mathsf{K}(x,x)\} = [(I-\mathsf{A}_x)\mathsf{K}(x,y)][\ {}_1\sigma_x\mathsf{K}(x,x)]$,

since $\mathbf{R}$ is the center of the Cayley-Dickson algebra $\mathcal{A}_r$ with $r\ge 2$, $<\mathsf{F}(z,y),\mathsf{K}(x,z),\ {}_1\sigma_x\mathsf{K}(x,x)>=0$ in these cases, where $<a,b,c>:=(ab)c-a(bc)$ denotes the associator of Cayley-Dickson numbers $a,b,c\in\mathcal{A}_r$. Then Conditions $7(1,2)$ imply that the function K satisfies the non-linear partial differential equation:

(6) $({}_1\sigma_x^2 - {}_2\sigma_y^2)\mathsf{K}(x,y) + 2\mathsf{p}\mathsf{K}(x,y)[\ {}_1\sigma_x\mathsf{K}(x,x)] = 0$.

If put $u(x) = 2\ {}_1\sigma_x\mathsf{K}(x,x)$ over the quaternion skew field $\mathbf{H}=\mathcal{A}_2$, i.e. for $r=2$, and substitute $\mathsf{K}(x,y)=\Phi(x,k)\exp(JRe(ky))$ into (6), we deduce that a function Φ satisfies Schrödinger's equation:

(7) ${}_1\sigma_x^2\Phi(x,k) + \Phi(x,k)(\mathsf{p}u + \sum_j k_j^2 i_j^2\ {}_2\psi_j^2) = 0$,

where $k\in\mathbf{H}$, since $\mathbf{H}$ is associative and the generator J commutes with $i_0,...,i_{2^r-1}$, also $\exp(JRe(ky))$ commutes with $\Phi(x,k)$.

Now we take the third order partial differential operator with ${}_1\psi_0 = {}_2\psi_0 = 0$:

(8) $L_2 f = ({}_3\sigma_t + {}_1\sigma_x^3 + 3\, {}_2\sigma_y\, {}_1\sigma_x^2 + 3\, {}_2\sigma_y^2\, {}_1\sigma_x + {}_2\sigma_y^3)f$. Then we put

(9) $L_{2,j} f = [{}_3\psi_{\xi_3(j)}\partial_{t_j} + ({}_1\psi_j\, \partial_{x_{\xi_1(j)}} + 3\, {}_2\psi_j \partial_{y_{\xi_2(j)}})\, {}_1\sigma_x^2 + ({}_2\psi_j\, \partial_{y_{\xi_2(j)}} + 3\, {}_1\psi_j \partial_{x_{\xi_1(j)}})\, {}_2\sigma_y^2] f$

and impose Condition 7(5) for $s = 2$:

(10) $L_{2,j}\mathsf{F}(x, y) = 0$ for each j,

where the operator ${}_3\sigma_t$ is with real constant coefficients ${}_3\psi_j$ and a transposition $\xi_3 \in S_{2^r}$. We suppose that functions F and K may depend on t. Therefore, we get from Equation (2) and Condition (10):

(11) $({}_3\sigma_t + {}_1\sigma_x^3 + 3\, {}_2\sigma_y\, {}_1\sigma_x^2 + 3\, {}_2\sigma_y^2\, {}_1\sigma_x + {}_2\sigma_y^3)\mathsf{K}(x, y)$

$= \mathsf{p}({}_3\sigma_t + {}_1\sigma_x^3 + 3\, {}_2\sigma_y\, {}_1\sigma_x^2 + 3\, {}_2\sigma_y^2\, {}_1\sigma_x + {}_2\sigma_y^3)\ {}_{1\sigma}\int_x^\infty \mathsf{F}(z, y)\mathsf{K}(x, z)dz =$

$-\mathsf{p}({}^1_1\sigma_z^3 + 3\, {}^1_2\sigma_y\, {}^1_1\sigma_z^2 + 3\, {}^1_2\sigma_y^2\, {}^1_1\sigma_z + {}^1_2\sigma_y^3)\ {}_{1\sigma}\int_x^\infty \mathsf{F}(z, y)\mathsf{K}(x, z)dz+$

$\mathsf{p}({}^2_3\sigma_t + {}_1\sigma_x^3 + 3\, {}_2\sigma_y\, {}_1\sigma_x^2 + 3\, {}_2\sigma_y^2\, {}_1\sigma_x + {}_2\sigma_y^3)\ {}_{1\sigma}\int_x^\infty \mathsf{F}(z, y)\mathsf{K}(x, z)dz =$

$I + \mathsf{p}\ {}^2_3\sigma_t\ {}_{1\sigma}\int_x^\infty \mathsf{F}(z, y)\mathsf{K}(x, z)dz$,

where $I = I_1 + I_2 + I_3$,

(12) $I_1 = \mathsf{p}({}_1\sigma_x^3 - {}^1_1\sigma_z^3)\ {}_{\sigma}\int_x^\infty \mathsf{F}(z, y)\mathsf{K}(x, z)dz$

$= \mathsf{p}({}^2_1\sigma_x^3 + {}^2_1\sigma_z^3)\ {}_{1\sigma}\int_x^\infty \mathsf{F}(z, y)\mathsf{K}(x, z)dz + \mathsf{p}\ {}_1A_3(\mathsf{F}, \mathsf{K})(x, y) - \mathsf{p}\ {}_1B_3(\mathsf{F}, \mathsf{K})(x, y)$

(13) $I_2 = 3\mathsf{p}({}_2\sigma_y\, {}_1\sigma_x^2 - {}_2\sigma_y\, {}^1_1\sigma_z^2)\ {}_{1\sigma}\int_x^\infty \mathsf{F}(z, y)\mathsf{K}(x, z)dz$

$= 3\mathsf{p}\ {}_2\sigma_y[({}^2_1\sigma_x^2 - {}^2_1\sigma_z^2)\ {}_{1\sigma}\int_x^\infty \mathsf{F}(z, y)\mathsf{K}(x, z)dz + {}_1A_2(\mathsf{F}, \mathsf{K})(x, y) - {}_1B_2(\mathsf{F}, \mathsf{K})(x, y)]$,

(14) $I_3 = 3\mathsf{p}({}_2\sigma_y^2\, {}_1\sigma_x - {}_2\sigma_y^2\, {}^1_1\sigma_z)\ {}_{1\sigma}\int_x^\infty \mathsf{F}(z, y)\mathsf{K}(x, z)dz$

$= 3\mathsf{p}\ {}_2\sigma_y^2({}^2_1\sigma_x + {}^2_1\sigma_z)\ {}_{1\sigma}\int_x^\infty \mathsf{F}(z, y)\mathsf{K}(x, z)dz$,

since ${}_1A_1 = {}_1B_1$. Equations $(4, 14)$, $4(3)$ and $5(1, 3)$ imply that

(15) $I_3 = 3\mathsf{p}\ {}^1_1\sigma_z^2({}^2_1\sigma_x + {}^2_1\sigma_z)\ {}_{1\sigma}\int_x^\infty \mathsf{F}(z, y)\mathsf{K}(x, z)dz$

$= 3\mathsf{p}({}^2_1\sigma_z - {}^1_1\sigma_x)\{({}^2_1\sigma_x + {}^2_1\sigma_z)[\mathsf{F}(x, y)\mathsf{K}(x, z)]\}|_{z=x} + 3\mathsf{p}({}^2_1\sigma_x\, {}^2_1\sigma_z^2 + {}^2_1\sigma_z^3)\ {}_{1\sigma}\int_x^\infty \mathsf{F}(z, y)\mathsf{K}(x, z)dz.$

Particularly, if the Cayley-Dickson algebra $\mathcal{A}_r$ is with $2 \le r \le 3$ and $\mathsf{F} \in Mat_s(\mathbf{R})$ and $\mathsf{K} \in Mat_s(\mathcal{A}_r)$, $s \in \mathbf{N}$ for $r = 2$, $s = 1$ for $r = 3$, then Equations $(6, 13)$ imply that

(16) $I_2 = -3\ {}_2\sigma_y[\mathsf{K}(x, y) - \mathsf{F}(x, y)]u(x) + 3\mathsf{p}\ {}_2\sigma_y[{}_1A_2(\mathsf{F}, \mathsf{K})(x, y) - {}_1B_2(\mathsf{F}, \mathsf{K})(x, y)]$,

since

(17) $-2\mathsf{p}\ {}_{1\sigma}\int_x^\infty \mathsf{F}(z, y)[\mathsf{K}(x, y)({}_1\sigma_x\mathsf{K}(x, x))]dz = -\mathsf{p}\ {}_{1\sigma}\int_x^\infty \mathsf{F}(z, y)[\mathsf{K}(x, z)u(x)]dz$

$= -\mathsf{p}[\ {}_{1\sigma}\int_x^\infty \mathsf{F}(z, y)\mathsf{K}(x, z)dz]u(x) = [\mathsf{K}(x, y) - \mathsf{F}(x, y)]u(x)$,

where $u(x) = 2\ {}_1\sigma_x\mathsf{K}(x, x)$. We deduce from Formulas $(6, 10, 11, 15-17)$ that

(18) $({}_3\sigma_t + {}_1\sigma_x^3 + 3\, {}_2\sigma_y\, {}_1\sigma_x^2 + 3\, {}_2\sigma_y^2\, {}_1\sigma_x + {}_2\sigma_y^3)\mathsf{K}(x, y) + 3{}^1_2\sigma_y[\mathsf{K}(x, y)u(x)] =$

$\mathsf{p}({}^2_3\sigma_t + {}^2_1\sigma_x^3 + 3\, {}^2_1\sigma_z\, {}^2_1\sigma_x^2 + 3\, {}^2_1\sigma_z^2\, {}^2_1\sigma_x + {}^2_1\sigma_z^3)\ {}_{1\sigma}\int_x^\infty \mathsf{F}(z, y)\mathsf{K}(x, z)dz$

$+3\mathsf{p}\ {}_{1\sigma}\int_x^\infty \mathsf{F}(z, y)[{}^1_2\sigma_z\mathsf{K}(x, z)u(x)]dz + T$, where

$T = \mathsf{p}\ {}_1A_3(\mathsf{F}, \mathsf{K})(x, y) - \mathsf{p}\ {}_1B_3(\mathsf{F}, \mathsf{K})(x, y) + 3\mathsf{p}\ {}_2\sigma_y[{}_1A_2(\mathsf{F}, \mathsf{K})(x, y) - {}_1B_2(\mathsf{F}, \mathsf{K})(x, y)] +$

$3\mathsf{p}\ {}^1_2\sigma_y[F(x, y)u(x)] + 3\mathsf{p}\ ({}^2_1\sigma_z\, {}^2_1\sigma_x + {}^2_1\sigma_z^2 - {}^1_1\sigma_x\, {}^2_1\sigma_x - {}^1_1\sigma_x\, {}^2_1\sigma_z)[\mathsf{F}(x, y)\mathsf{K}(x, z)]|_{z=x}.$

Then for each two continuously differentiable $\mathcal{A}_r$ valued functions $\mathsf{G}(z, y)$ and $\mathsf{K}(x, z)$ one has

$({}^1\sigma_z - {}^2\sigma_x - {}^2\sigma_z)[\mathsf{G}(z, y)\mathsf{K}(x, z)] = ({}^1\sigma_z + {}^2\sigma_x + {}^2\sigma_z)[\mathsf{G}(z, y)\check{\mathsf{K}}(x, z)] =$

$(\sigma_x + \sigma_z)[\mathsf{G}(z, y)\check{\mathsf{K}}(x, z)]$,

where $\check{\mathsf{K}}(x,z) := \mathsf{K}(-x,-z)$ for each $-x,-z \in U$. Therefore, the identity

$[_\sigma \int_x, [\ ^1\sigma_x,\ ^2\sigma_x]]\{\mathsf{G}(z,y)\mathsf{K}(x,z)\}dz = 0$

is satisfied, consequently, in the considered case $2 \le r \le 3$ and $\mathsf{F} \in Mat_s(\mathbf{R})$ and $\mathsf{K} \in Mat_s(\mathcal{A}_r)$, $s \in \mathbf{N}$ for $r=2$, $s=1$ for $r=3$, we get

(19) $[(I-A_x),[^1\sigma_x,{}^2\sigma_x]]\{\mathsf{F}(x,y)\mathsf{K}(x,x)\} = 0,$

since $\mathbf{R}$ is the center of the Cayley-Dickson algebra $\mathcal{A}_r$ and

(20) $[^2\sigma_x,\ ^3\sigma_x]\mathsf{F}(z,y)\{\mathsf{F}(x,z)\mathsf{K}(x,x)\} = \mathsf{F}(z,y)([^1\sigma_x,\ ^2\sigma_x]\{\mathsf{F}(x,z)\mathsf{K}(x,x)\})$

due to Formulas (1), 4(7,8) and 7(2), since $[^1\sigma_x +\ ^2\sigma_x, {}^1\sigma_x -\ ^2\sigma_x] = -2[^1\sigma_x,\ ^2\sigma_x]\{\mathsf{G}(x,y)\mathsf{K}(x,x)\}$.

Substituting the expressions of $A_2 - B_2$ and $A_3 - B_3$ from Corollary 5 and using Formulas $(2,6,19,20)$ one gets:

(21) $T = -\mathsf{p}(3\ {}^2_1\sigma_x^2 +\ {}^2_1\sigma_x\ {}^2_1\sigma_z + 2\ {}^2_1\sigma_z\ {}^2_1\sigma_x)[\mathsf{F}(x,y)\mathsf{K}(x,z)]|_{z=x}$

$-\mathsf{p}(2\ {}^1_1\sigma_x\ {}^2_1\sigma_x +\ {}^2_1\sigma_x\ {}^1_1\sigma_x)[\mathsf{F}(x,y)\mathsf{K}(x,x)]$

$+3(1-\mathsf{p})\ {}_2\sigma_y[\mathsf{F}(x,y)u(x)] + 3\mathsf{p}\ (\ {}^2_1\sigma_z\ {}^2_1\sigma_x + {}^2_1\sigma_z^2 - {}^1_1\sigma_x\ {}^2_1\sigma_x - {}^1_1\sigma_x\ {}^2_1\sigma_z)[\mathsf{F}(x,y)\mathsf{K}(x,z)]|_{z=x}$

$= -3\mathsf{p}\ {}^1_1\sigma_x[\mathsf{F}(x,y)u(x)] - 3\mathsf{p}({}^2_1\sigma_x^2 - {}^2_1\sigma_z^2)[\mathsf{F}(x,y)\mathsf{K}(x,z)]|_{z=x} + 3(1-\mathsf{p})\ {}_2\sigma_y[\mathsf{F}(x,y)u(x)]$

$+\mathsf{p}[{}^2_1\sigma_z,\ {}^2_1\sigma_x][\mathsf{F}(x,y)\mathsf{K}(x,z)]|_{z=x} + \mathsf{p}[{}^1_1\sigma_x,\ {}^2_1\sigma_x][\mathsf{F}(x,y)\mathsf{K}(x,x)]$

$= -3\mathsf{p}\ {}^1_1\sigma_x[\mathsf{F}(x,y)u(x)] + 3\mathsf{p}\mathsf{F}(x,y)[\mathsf{K}(x,x)u(x)] + 3(1-\mathsf{p})\ {}_2\sigma_y[\mathsf{F}(x,y)u(x)]$

$+\mathsf{p}[{}^2_1\sigma_z,\ {}^2_1\sigma_x][\mathsf{F}(x,y)\mathsf{K}(x,z)]|_{z=x} + \mathsf{p}[{}^1_1\sigma_x,\ {}^2_1\sigma_x][\mathsf{F}(x,y)\mathsf{K}(x,x)]$

$= -3\mathsf{p}\ {}^1_1\sigma_x[\mathsf{K}(x,y)u(x)] + 3\mathsf{p}^2\ {}_1\sigma_x\{({}_{1\sigma}\int_x^\infty \mathsf{F}(z,y)\mathsf{K}(x,z)dz)u(\eta)\}|_{\eta=x} + 3\mathsf{p}\mathsf{F}(x,y)[\mathsf{K}(x,x)u(x)]$

$+3(1-\mathsf{p})\ {}_2\sigma_y[\mathsf{F}(x,y)u(x)] + \mathsf{p}[{}^2_1\sigma_z,\ {}^2_1\sigma_x][\mathsf{F}(x,y)\mathsf{K}(x,z)]|_{z=x} + \mathsf{p}[{}^1_1\sigma_x,\ {}^2_1\sigma_x][\mathsf{F}(x,y)\mathsf{K}(x,x)]$

$= -3\mathsf{p}\ {}^1_1\sigma_x[\mathsf{F}(x,y)u(x)] + 3\mathsf{p}^2\ {}^2_1\sigma_x[({}_{1\sigma}\int_x^\infty[(\mathsf{F}(z,y)\mathsf{K}(x,z))u(x)]dz$

$+3(1-\mathsf{p})\ {}_2\sigma_y[(I-A_x)\mathsf{K}(x,y)u(x)] + \mathsf{p}[(I-A_x)\mathsf{K}(x,y)]\{[{}_1\sigma_z,\ {}_1\sigma_x]\mathsf{K}(x,z)]|_{z=x}\} +$

$\mathsf{p}(I-A_x)\{[{}^1_1\sigma_x,\ {}^2_1\sigma_x][\mathsf{K}(x,y)\mathsf{K}(x,x)]\},$

since $2 \le r \le 3$, $F \in Mat_s(\mathbf{R})$ and $\mathsf{K} \in Mat_s(\mathcal{A}_r)$ with $s \in \mathbf{N}$ for $r=2$ and $s=1$ for $r=3$, hence $< \mathsf{F}(x,y), \mathsf{K}(x,x), u(x) >= 0$, where $< a,b,c > := (ab)c - a(bc)$ denotes the associator for each Cayley-Dickson numbers $a,b,c \in \mathcal{A}_r$, since $i_0 i_k - i_k i_0 = 0$, $i_j i_k = -i_k i_j$ for each $j \ne k \ge 1$,

(22) $[\sigma_z, \sigma_x] = (\sigma_z\sigma_x - \sigma_x\sigma_z) = -[\sigma_x,\sigma_z].$

We now take $\mathsf{p}=1$. Therefore, in accordance with Formulas $(19-21)$ and 7(13) the equality

(23) $({}_3\sigma_t +\ {}_1\sigma_x^3 +\ 3\ {}_2\sigma_y\ {}_1\sigma_x^2 +\ 3\ {}_2\sigma_y^2\ {}_1\sigma_x +\ {}_2\sigma_y^3)\mathsf{K}(x,y)$

$+6({}^1_1\sigma_x +\ {}^1_2\sigma_y)[\mathsf{K}(x,y)(\ {}_1\sigma_x\mathsf{K}(x,x))] - \mathsf{K}(x,y)\{[{}_1\sigma_z,\ {}_1\sigma_x]\mathsf{K}(x,z)]|_{z=x}\}$

$-[{}^1_1\sigma_x,\ {}^2_1\sigma_x][\mathsf{K}(x,y)\mathsf{K}(x,x)] = 0$

follows, when the operator $(I-A_x)$ is invertible. Particularly, for $s=1$ and ${}_1\sigma = {}_2\sigma$ with ${}_1\psi_0 = 0$ and ${}_3\sigma_t = \partial/\partial t_0$ differentiating Equation (23) with the operator ${}_1\sigma_x$ and then restricting on the diagonal $x=y$ and taking into account Formulas $(1,6)$, 7(1) and (19) one gets the equation

(24) $u_t(t,x) + 6\ {}_1\sigma_x[u(t,x)u(t,x)] +\ {}_1\sigma_x^3 u(t,x) = 0$

of Korteweg-de-Vries' type, where $u(t,x) = 2\ {}_1\sigma_x\mathsf{K}(x,x)$, since $\sigma(f(x)g(x)) =$

$({}^1\sigma + {}^2\sigma)(f(x)g(x))$, $[{}^1\sigma, {}^2\sigma](f(x)g(x)) = -\frac{1}{2}[\sigma, {}^1\sigma - {}^2\sigma](f(x)g(x))$, $\sigma[{}^1\sigma, {}^2\sigma](f(x)g(x)) = -[{}^1\sigma, {}^2\sigma]\sigma(f(x)g(x))$ and hence $\sigma^{-1}[{}^1\sigma, {}^2\sigma](f(x)g(x)) = -[{}^1\sigma, {}^2\sigma]\sigma^{-1}(f(x)g(x))$ for $\sigma = {}_1\sigma_x$ with ${}_1\psi_0 = 0$ and continuously differentiable functions $f(x)$ and $g(x)$ (see Remark 3.2).

8.1. Theorem. *A solution of partial differential Equation* (23) *with* ${}_1\psi_0 = {}_2\psi_0 = 0$ *over the Cayley-Dickson algebra* $\mathcal{A}_r$ *with* $2 \le r \le 3$ *is given by Formulas* $(2-4, 9, 10)$ *with* $\mathsf{p} = 1$ *whenever the appearing integrals uniformly converge by parameters on compact sub-domains (see Proposition 4) and the operator* $(I - A_x)$ *is invertible and* $\mathsf{F} \in Mat_s(\mathbf{R})$ *and* $\mathsf{K} \in Mat_s(\mathcal{A}_r)$, $s \in \mathbf{N}$ *when* $r = 2$, $s = 1$ *when* $r = 3$.

9. Example. Consider the integral equation

(1) $\mathsf{K}(x,y) = \mathsf{F}(x,y) + \frac{\mathsf{p}}{4}\, {}_\sigma\int_x^\infty (\, {}_\sigma\int_x^\infty \mathsf{F}(u,y)[\mathsf{F}(z,u)\mathsf{K}(x,z)]dz)du$,

where p is a non-zero real constant. We take, for example,

(2) $L_1 = \sigma_x - {}_1\sigma_y$ with $\psi_0 = {}_1\psi_0 = 0$.

A solution of the equation

(3) $L_1\mathsf{F}(x,y) = 0$

has the form $\mathsf{F}(x,y) = \mathsf{G}(\frac{(a,x>+({}_1a,y>}{2})$ or we shall write $\mathsf{F}(\frac{(a,x>+({}_1a,y>}{2})$ instead of $\mathsf{F}(x,y)$, where

(4) $(a, x >:= \sum_{j=0}^{2^r-1} a_j x_j i_j$

for $a, x \in \mathcal{A}_r$, $x = \sum_{j=0}^{2^r-1} x_j i_j$, $x_j \in \mathbf{R}$ for each j, $a_j = \psi_j$ for $\psi_j \ne 0$ and $a_j = 1$ for $\psi_j = 0$. Using suitable change of real variables x_j, y_j we can suppose without loss of generality, that ψ_j, ${}_1\psi_j \in \{0,1\}$ for each $j = 0, ..., 2^r - 1$. Therefore, a solution of Equation (3) can be written as

(3.1) $\mathsf{F}(\frac{(b,x+y>}{2})$ as well, where $b_j = a_j\, {}_1a_j \in \{0,1\}$ for each j. Differentiable functions of the form $\mathsf{F}(\frac{(b,x+y>}{2})$ are also solutions of the system of partial differential equations

(3.2) $L_{1,j}\mathsf{F}(\frac{(b,x+y>}{2}) = 0$, where

(3.3) $L_{1,j} = \psi_j\partial_{x_j} - {}_1\psi_j\partial_{y_j}$.

It is convenient to introduce the notation

(5) $\mathsf{K}_2(x,z) := {}_\sigma\int_0^\infty \mathsf{F}(\frac{(a,x+\zeta>+({}_1a,z>}{2})\mathsf{K}(x,x+\zeta)d\zeta$.

Using Condition (3) we rewrite Equation (1) in the form:

(6) $\mathsf{K}(x,y) = \mathsf{F}(\frac{(a,x>+({}_1a,y>}{2})$
$+\frac{\mathsf{p}}{4}\, {}_\sigma\int_0^\infty {}_\sigma\int_0^\infty [\mathsf{F}(\frac{(a,x+\eta>+({}_1a,y>}{2})(\mathsf{F}(\frac{(a,x+\zeta>+({}_1a,x+\eta>}{2})\mathsf{K}(x,x+\zeta))d\zeta]d\eta$
$= \mathsf{F}(\frac{(a,x>+({}_1a,y>}{2}) + \frac{\mathsf{p}}{4}\, {}_\sigma\int_0^\infty \mathsf{F}(\frac{(a,x+\eta>+({}_1a,y>}{2})\mathsf{K}_2(x,x+\eta)d\eta$.

Then let us put:

(7) $A_xf(y) := \frac{\mathsf{p}}{4}\, {}_\sigma\int_0^\infty {}_\sigma\int_0^\infty [\mathsf{F}(\frac{(a,x+\eta>+({}_1a,y>}{2})(\mathsf{F}(\frac{(a,x+\zeta>+({}_1a,x+\eta>}{2})f(x+\zeta))d\zeta]d\eta$

for a continuous function f, consequently,

(8) $(I - A_x)\mathsf{K}(x,y) = \mathsf{F}(\frac{(a,x>+({}_1a,y>}{2})$.

Particularly, if $\mathsf{F} \in Mat_s(\mathbf{R})$ and $\mathsf{K} \in Mat_s(\mathcal{A}_r)$, then

(9) $(I - A_x)\mathsf{K}_2(x,y) = \ {}_{\sigma}\int_0^\infty \mathsf{F}(\frac{(a,x+\zeta>+(_1a,y>}{2})\mathsf{F}(\frac{(a,x>+(_1a,x+\zeta>}{2})d\zeta.$

Acting on both sides of Equation (1) with the operator L_1 leads to the relation:

(10) $(\sigma_x - \ _1\sigma_y)\mathsf{K}(x,y) = \frac{\mathsf{p}}{4} \ {}^2\sigma_x \ {}_{\sigma}\int_0^\infty \mathsf{F}(\frac{(a,x+\eta>+(_1a,y>}{2})\mathsf{K}_2(x,x+\eta)d\eta.$

Now we deduce an expression for $\sigma_x\mathsf{K}_2(x,x+\eta)$. From the definition of K_2 the identity

(11) $(\sigma_x + \ _1\sigma_z)\mathsf{K}_2(x,z) = ({}^1\sigma_x + \ {}^1_1\sigma_z) \ {}_{\sigma}\int_0^\infty \mathsf{F}(\frac{(a,x+\zeta>+(_1a,z>}{2})\mathsf{K}(x,x+\zeta)d\zeta$
$+{}^2\sigma_x \ {}_{\sigma}\int_0^\infty \mathsf{F}(\frac{(a,x+\zeta>+(_1a,z>}{2})\mathsf{K}(x,x+\zeta)d\zeta$

follows. Using (3.1) we get that

(12) $(\sigma_x + {}_1\sigma_z)\mathsf{F}(\frac{(a,x+\zeta>+(_1a,z>}{2}) = (\sigma_x + {}_1\sigma_z)\mathsf{F}(\frac{(b,x+\zeta+z>}{2}) = \sigma_\xi \mathsf{F}(\xi)|_{\xi=\frac{(b,x+\zeta+z>}{2}},$

since $b_ja_j = b_j$ and $b_j \ _1a_j = b_j$ for each $j = 0, ..., 2^r - 1$. Therefore, in accordance with Proposition 4 the equality

(13) $(\sigma_x + \sigma_z)\mathsf{K}_2(x,z) = ({}^2\sigma_1 - \ {}^2\sigma_2) \ {}_{\sigma}\int_0^\infty \mathsf{F}(\frac{(b,x+\zeta+z>}{2})\mathsf{K}(x,x+\zeta)d\zeta - 2\mathsf{F}(\frac{(b,x+z>}{2})\mathsf{K}(x,x)$

follows, where $\sigma_1\mathsf{K}(x,z) := \sigma_x\mathsf{K}(x,z)$ and $\sigma_2\mathsf{K}(x,z) := \sigma_z\mathsf{K}(x,z)$. We seek a solution $\mathsf{K}(x,y)$ satisfying the condition:

(14) $\sigma_y\mathsf{K}(x,y) = \ _1\sigma_y\mathsf{K}(x,y).$

From Equations $(10, 13, 14)$ we get, that

(15) $(\sigma_x + \sigma_z)\mathsf{K}_2(x,z) = -2\mathsf{F}(\frac{(b,x+z>}{2})\mathsf{K}(x,x)$
$+\frac{\mathsf{p}}{4} \ {}^3\sigma_x \ {}_{\sigma}\int_0^\infty \mathsf{F}(\frac{(b,x+\zeta+z>}{2})\{ \ {}_{\sigma}\int_0^\infty[\mathsf{F}(\frac{(b,2x+\eta+\zeta>}{2})\mathsf{K}_2(x,x+\eta)]d\eta\}d\zeta$, consequently,

(16) $(I-A_x)[(\sigma_x+\sigma_z)\mathsf{K}_2(x,z)] = -2\mathsf{F}(\frac{(b,x+z>}{2})\mathsf{K}(x,x) = -2[(I-A_x)\mathsf{K}(x,z)]\mathsf{K}(x,x).$

If either $r = 2$ or $\mathsf{F} \in Mat_s(\mathbf{R})$ and $\mathsf{K} \in Mat_s(\mathcal{A}_r)$, then one obtains from $(3.3, 5, 6, 13)$ analogously the identities:

(17) $(\sigma_x - {}_1\sigma_y)\mathsf{K}(x,y) = \frac{\mathsf{p}}{4}({}^3\sigma_1 - {}^3\sigma_2) \ {}_{\sigma}\int_0^\infty \mathsf{F}(\frac{(b,x+\zeta+y>}{2})\{ \ {}_{\sigma}\int_0^\infty[\mathsf{F}(\frac{(a,2x+\eta+\zeta>}{2})\mathsf{K}(x,x+\zeta)]d\zeta - 2\mathsf{F}(\frac{(a,2x+\eta>}{2})\mathsf{K}(x,x)\}d\eta$
$= A_x[(\sigma_x - \sigma_y)\mathsf{K}(x,y)] - \frac{\mathsf{p}}{2} \ {}_{\sigma}\int_0^\infty \mathsf{F}(\frac{(b,x+\zeta+y>}{2})[\mathsf{F}(\frac{(a,2x+\eta>}{2})\mathsf{K}(x,x)]d\eta.$

Therefore,

(18) $(I - A_x)[(\sigma_x - \sigma_y)\mathsf{K}(x,y)] = -\frac{\mathsf{p}}{2}[(I - A_x)\mathsf{K}_2(x,y)]\mathsf{K}(x,x) + \frac{\mathsf{p}}{2} \ {}_{\sigma}\int_0^\infty < \mathsf{F}(\frac{(b,x+\eta+y>}{2}), \ \mathsf{F}(\frac{(b,2x+\eta>}{2}), \ \mathsf{K}(x,x) > d\eta.$

In the associative quaternion case $\mathcal{A}_2 = \mathbf{H}$ or when $\mathsf{F} \in Mat_s(\mathbf{R})$ and $\mathsf{K} \in Mat_s(\mathcal{A}_r)$ the last additive in Formula (18) vanishes, i.e.

(18.1) $(I - A_x)[(\sigma_x - \sigma_y)\mathsf{K}(x,y)] = -\frac{\mathsf{p}}{2}[(I - A_x)\mathsf{K}_2(x,y)]\mathsf{K}(x,x).$

On the other hand, one has the identities:

(19) $\mathsf{K}_2(x,x+\zeta) = {}_{\sigma}\int_0^\infty \mathsf{F}(\frac{(b,2x+\xi+\zeta>}{2})\mathsf{K}(x,x+\xi)d\xi$ and

(19.1) $(I - A_x)\mathsf{K}_2(x,z) = {}_{\sigma}\int_0^\infty \mathsf{F}(\frac{(b,x+\zeta+z>}{2})\mathsf{K}(x,x+\zeta)d\zeta$
$-\frac{\mathsf{p}}{4} \ {}_{\sigma}\int_0^\infty \mathsf{F}(\frac{(b,x+z+\eta>}{2})\{\mathsf{F}(\frac{(b,2x+\eta+\zeta>}{2})[\ {}_{\sigma}\int_0^\infty \mathsf{F}(\frac{(b,2x+\xi+\zeta>}{2})\mathsf{K}(x,x+\xi)d\xi]d\zeta\}d\eta =$
${}_{\sigma}\int_0^\infty \mathsf{F}(\frac{(b,x+z+\zeta>}{2})[(I - A_x)\mathsf{K}(x,x+\zeta)]d\zeta$
$= \ {}_{\sigma}\int_0^\infty \mathsf{F}(\frac{(b,x+z+\zeta>}{2})\mathsf{F}(\frac{(b,2x+\zeta>}{2})d\zeta$, that is

(20) $(I-A_x)\mathsf{K}_2(x,z) = {}_\sigma\int_0^\infty \mathsf{F}(\frac{(b,x+z+\zeta>}{2})\mathsf{F}(\frac{(b,2x+\zeta>}{2})d\zeta$.

If either $r=2$, i.e. $\mathcal{A}_2=\mathbf{H}$, or when $\mathsf{F}\in Mat_s(\mathbf{R})$ and $\mathsf{K}\in Mat_s(\mathcal{A}_r)$, $s\in\mathbf{N}$ for $r=2$, $s=1$ for $r=3$, Equations $(16,18.1)$ imply that the functions K and K_2 are solutions of the system of partial differential equations:

(21) $(\sigma_x+\sigma_z)\mathsf{K}_2(x,z) = -2\mathsf{K}(x,z)\mathsf{K}(x,x)$ and

(22) $(\sigma_x-\sigma_z)\mathsf{K}(x,z) = -\frac{\mathsf{p}}{2}\mathsf{K}_2(x,z)\mathsf{K}(x,x)$.

The action on both sides of Equation (1) with the operator $\sigma_x + {}_1\sigma_y$ leads to the identities:

(23) $(\sigma_x + {}_1\sigma_y)\mathsf{K}(x,y) = \sigma_z F(z)|_{z=(b,x+y>/2}$
$+\frac{\mathsf{p}}{4}\,{}^3\sigma_x\,{}_\sigma\int_0^\infty {}_\sigma\int_0^\infty\{\mathsf{F}(\frac{(b,x+\eta+y>}{2})[\mathsf{F}(\frac{(b,2x+\zeta+\eta>}{2})\mathsf{K}(x,x+\zeta)]d\zeta\}d\eta$
$-\frac{\mathsf{p}}{2}\mathsf{F}(\frac{(b,x+y>}{2})\,{}_\sigma\int_0^\infty[\mathsf{F}(\frac{(b,2x+\zeta>}{2})\mathsf{K}(x,x+\zeta)]d\zeta$,

since

$({}^1\sigma_x + {}^1_1\sigma_y + {}^2\sigma_x + {}^2_1\sigma_y)\{\mathsf{F}(\frac{(b,x+\eta+y>}{2})[\mathsf{F}(\frac{(b,2x+\zeta+\eta>}{2})g(x,\zeta)]\}$
$=2\sigma_\eta\{\mathsf{F}(\frac{(b,x+\eta+y>}{2})[\mathsf{F}(\frac{(b,2x+\zeta+\eta>}{2})g(x,\zeta)]\}$

for each $\mathcal{A}_r$ valued function $g(x,\zeta)$, consequently,

(24) $(I-A_x)[(\sigma_x + {}_1\sigma_y)\mathsf{K}(x,y)] = \sigma_z F(z)|_{z=(b,x+y>/2} - \frac{\mathsf{p}}{2}[(I-A_x)\mathsf{K}(x,y)]\mathsf{K}_2(x,x)$.

Then we use the condition

(25) $L_{2,j}\mathsf{F}=0$ for each $j=0,...,2^r-1$,

where

(26) $L_{2,j} = {}_2\psi_j\partial_{t_j} + \psi_j\partial_{x_j}\sigma_x^2 + 3\,{}_1\psi_j\partial_{y_j}\sigma_x^2 + 3\psi_j\,{}_1\sigma_y^2\partial_{x_j} + {}_1\psi_j\partial_{y_j}\,{}_1\sigma_y^2$

and act on both sides of Equation (1) with the operator

$L_2 = {}_2\sigma_t + \sigma_x^3 + 3\,{}_1\sigma_y\sigma_x^2 + 3\,{}_1\sigma_y^2\sigma_x + {}_1\sigma_y^3$ with $\psi_0=0$, ${}_1\psi_0=0$, that gives

(27) $L_2\mathsf{K}(x,y) = \frac{\mathsf{p}}{4}({}_2\sigma_t + \sigma_x^3 + 3\,{}_1\sigma_y\sigma_x^2 + 3\,{}_1\sigma_y^2\sigma_x + {}_1\sigma_y^3)$
${}_\sigma\int_0^\infty {}_\sigma\int_0^\infty\{\mathsf{F}(\frac{(b,x+\eta+y>}{2})[\mathsf{F}(\frac{(b,2x+\zeta+\eta>}{2})\mathsf{K}(x,x+\zeta)]d\zeta\}d\eta$.

To simplify appearing formulas we use the identity

(28) $({}^1L_2 + {}^2L_2)\{\mathsf{F}(\frac{(b,x+\eta+y>}{2})[\mathsf{F}(\frac{(b,2x+\zeta+\eta>}{2})g(x,\xi)]\} =$
$\{({}^1\sigma_x + {}^2\sigma_x)[{}^1\sigma_x\,{}^2\sigma_x + {}^2\sigma_x\,{}^1\sigma_x] + 3{}^1_1\sigma_y^2\,{}^2\sigma_x + 3{}^1_1\sigma_y({}^1\sigma_x\,{}^2\sigma_x + {}^2\sigma_x\,{}^1\sigma_x + {}^2\sigma_x^2)\}$
$\{\mathsf{F}(\frac{(b,x+\eta+y>}{2})[\mathsf{F}(\frac{(b,2x+\zeta+\eta>}{2})g(x,\xi)]\}$
$=3\sigma_\eta({}^2\sigma_z\,{}^1\sigma_\zeta + {}^1\sigma_\zeta\,{}^2\sigma_z)\{\mathsf{F}(\zeta)|_{\zeta=\frac{(b,x+\eta+y>}{2}}[\mathsf{F}(z)|_{z=\frac{(b,2x+\zeta+\eta>}{2}}g(x,\xi)]\}$

for every $\mathcal{A}_r$ valued function $g(x,\xi)$, since the function F satisfies Conditions (25) and

(28.1) $\sigma_x\,{}_1\sigma_y^2[f(x,y)g(x,y)] = ({}^1\sigma_x + {}^2\sigma_x)[({}^1_1\sigma_y\,{}^2_1\sigma_y + {}^2_1\sigma_y\,{}^1_1\sigma_y) + ({}^1_1\sigma_y^2 + {}^2_1\sigma_y^2)][f(x,y)g(x,y)]$

and

(28.2) $\sigma_x^3[f(x,y)g(x,y)] = ({}^1\sigma_x + {}^2\sigma_x)[({}^1\sigma_x\,{}^2\sigma_x + {}^2\sigma_x\,{}^1\sigma_x) + ({}^1\sigma_x^2 + {}^2\sigma_x^2)][f(x,y)g(x,y)]$

for each $\mathcal{A}_r$ valued twice differentiable functions f and g.

Then we have also from $(3,5)$:

(29) $(\sigma_x - \sigma_y)\mathsf{K}_2(x,y) = ({}^1\sigma_x - {}^1\sigma_y + {}^2\sigma_x)\,{}_\sigma\int_0^\infty[\mathsf{F}(\frac{(b,x+\zeta+y>}{2})\mathsf{K}(x,x+\zeta)]d\zeta$
$= {}^2\sigma_x\,{}_\sigma\int_0^\infty[\mathsf{F}(\frac{(b,x+\zeta+y>}{2})\mathsf{K}(x,x+\zeta)]d\zeta$,

since $\partial_{x_j}\mathsf{F}(x,y) = \partial_{y_j}\mathsf{F}(x,y)$ for each x,y and j.

Hence, the identity is satisfied:

(30) $\sigma_x\ {}^2\sigma_x[\ {}_\sigma\int_0^\infty \mathsf{F}(\frac{(b,2x+\zeta>}{2})\mathsf{K}(x,x+\zeta)d\zeta] = -2\ {}^2\sigma_x[\mathsf{K}(x,x)\mathsf{K}(x,x)] - 2(\ {}^1\sigma_x -\ {}^1\sigma_y)[\mathsf{K}(x,y)\mathsf{K}(x,x)]|_{y=x} - 2[\sigma_x,\sigma_y]\mathsf{K}_2(x,y)|_{y=x}$,

since $[(\sigma_x+\sigma_y),(\sigma_x-\sigma_y)] = -2[\sigma_x,\sigma_y]$ and

$\sigma_x\ {}^2\sigma_x[\ {}_\sigma\int_0^\infty \mathsf{F}(\frac{(b,2x+\zeta>}{2})\mathsf{K}(x,x+\zeta)d\zeta] = \sigma_x\{[(\sigma_x-\sigma_y)\mathsf{K}_2(x,y)]|_{y=x}\}$

$= (\sigma_x+\sigma_y)(\sigma_x-\sigma_y)\mathsf{K}_2(x,y)|_{y=x}$

$= (\sigma_x-\sigma_y)(\sigma_x+\sigma_y)\mathsf{K}_2(x,y)|_{y=x} - 2[\sigma_x,\sigma_y]\mathsf{K}_2(x,y)|_{y=x}$

$= (\sigma_x-\sigma_y)[-2\mathsf{K}(x,y)\mathsf{K}(x,x)]|_{y=x} - 2[\sigma_x,\sigma_y]\mathsf{K}_2(x,y)|_{y=x}$

due to Identity (16). On the other hand,

$[\sigma_x,\sigma_y]\mathsf{K}_2(x,y) = ({}^1\sigma_x\ {}^1\sigma_y -\ {}^1\sigma_y\ {}^1\sigma_x +\ {}^2\sigma_x\ {}^1\sigma_y -\ {}^1\sigma_y\ {}^2\sigma_x)\ {}_\sigma\int_0^\infty[\mathsf{F}(\frac{(b,x+\zeta+y>}{2})\mathsf{K}(x,x+\zeta)]d\zeta$

$= (\ {}^2\sigma_x\ {}^1\sigma_y -\ {}^1\sigma_y\ {}^2\sigma_x)\ {}_\sigma\int_0^\infty[\mathsf{F}(\frac{(b,x+\zeta+y>}{2})\mathsf{K}(x,x+\zeta)]d\zeta$,

since $({}^1\sigma_x\ {}^1\sigma_y -\ {}^1\sigma_y\ {}^1\sigma_x)[\mathsf{F}(\frac{(b,x+\zeta+y>}{2})\mathsf{K}(x,x+\zeta)] = 0$, consequently,

$[\sigma_x,\sigma_y]\mathsf{K}_2(x,y)|_{y=x} = \frac{1}{2}(\ {}^2\sigma_x\ {}^1\sigma_x -\ {}^1\sigma_x\ {}^2\sigma_x)\ {}_\sigma\int_0^\infty[\mathsf{F}(\frac{(b,2x+\zeta>}{2})\mathsf{K}(x,x+\zeta)]d\zeta$

and

(31) $(\sigma_x\ {}^2\sigma_x +\ {}^2\sigma_x\sigma_x)[\ {}_\sigma\int_0^\infty \mathsf{F}(\frac{(b,2x+\zeta>}{2})\mathsf{K}(x,x+\zeta)d\zeta] = -4\ {}^2\sigma_x[\mathsf{K}(x,x)\mathsf{K}(x,x)] - 4(\ {}^1\sigma_x -\ {}^1\sigma_y)[\mathsf{K}(x,y)\mathsf{K}(x,x)]|_{y=x}$.

Using Equations $(28-28.2)$ we rewrite Equation (27) in accordance with $4(9-11)$ in the form:

(32) $(I-A_x)[L_2\mathsf{K}(x,y)] =$

$\frac{\mathsf{p}}{4}(L_2 -\ {}^3L_2)\ {}_\sigma\int_0^\infty\ {}_\sigma\int_0^\infty\{\mathsf{F}(\frac{(b,x+y+\eta>}{2})[\mathsf{F}(\frac{(b,2x+\zeta+\eta>}{2})\mathsf{K}(x,x+\zeta)]d\eta\}d\zeta$

$+\frac{3\mathsf{p}}{4}[({}^1\sigma_x\ {}^3\sigma_x +\ {}^3\sigma_x\ {}^1\sigma_x +\ {}^2\sigma_x\ {}^3\sigma_x +\ {}^3\sigma_x\ {}^2\sigma_x)\ {}^1_1\sigma_y +\ {}^1_1\sigma_y^2\ {}^3\sigma_x + ({}^1\sigma_x\ {}^2\sigma_x +\ {}^2\sigma_x\ {}^1\sigma_x +\ {}^1\sigma_x^2 +\ {}^2\sigma_x^2)\ {}^3\sigma_x]$

$(\ {}_\sigma\int_0^\infty\ {}_\sigma\int_0^\infty\{\mathsf{F}(\frac{(b,x+y+\eta>}{2})[\mathsf{F}(\frac{(b,2x+\zeta+\eta>}{2})\mathsf{K}(x,x+\zeta)]d\eta\}d\zeta$

$+\frac{3\mathsf{p}}{4}[({}^1\sigma_x +\ {}^2\sigma_x +\ {}^1\sigma_y)\ {}^3\sigma_x^2(\ {}_\sigma\int_0^\infty\ {}_\sigma\int_0^\infty\{\mathsf{F}(\frac{(b,x+y+\eta>}{2})[\mathsf{F}(\frac{(b,2x+\zeta+\eta>}{2})$

$\mathsf{K}(x,x+\zeta)]d\eta\}d\zeta$

$= -\frac{3\mathsf{p}}{4}[{}^1\sigma_z\ {}^2\sigma_\xi +\ {}^2\sigma_\xi\ {}^1\sigma_z]\ {}_\sigma\int_0^\infty \mathsf{F}(z)|_{z=\frac{(b,x+y>}{2}}[\mathsf{F}(\xi)|_{\xi=\frac{(b,2x+\zeta>}{2}}\mathsf{K}(x,x+\zeta)]d\zeta$

$-\frac{3\mathsf{p}}{2}\ {}^3\sigma_x^2\ {}_\sigma\int_0^\infty \mathsf{F}(\frac{(b,x+y>}{2})[\mathsf{F}(\frac{(b,2x+\zeta>}{2})\mathsf{K}(x,x+\zeta)]d\zeta$

$+3\mathsf{p}\ {}^3\sigma_x\sigma_\eta^2\ {}_\sigma\int_0^\infty\ {}_\sigma\int_0^\infty\{\mathsf{F}(\frac{(b,x+\eta+y>}{2})[\mathsf{F}(\frac{(b,2x+\zeta+\eta>}{2})\mathsf{K}(x,x+\zeta)]d\eta\}d\zeta$

since

$(\sigma_x^2 +\ {}_1\sigma_y^2)[\mathsf{F}(\frac{(b,x+\eta+y>}{2})\mathsf{F}(\frac{(b,2x+\zeta+\eta>}{2})] = 4\sigma_\eta^2[\mathsf{F}(\frac{(b,x+\eta+y>}{2})\mathsf{F}(\frac{(b,2x+\zeta+\eta>}{2})]$.

If $\mathsf{F}\in Mat_s(\mathbf{R})$ and $\mathsf{K}\in Mat_s(\mathcal{A}_r)$, $2\le r\le 3$, $s\in\mathbf{N}$ for $r=2$, $s=1$ for $r=3$, then using Equation (11) we deduce that

(33) $(I-A_x)[L_2\mathsf{K}(x,y)] = -\frac{3\mathsf{p}}{4}({}^1\sigma_\eta\ {}^2\sigma_x +\ {}^2\sigma_x\ {}^1\sigma_\eta)[\mathsf{F}(\eta)|_{\eta=\frac{x+y}{2}}\mathsf{K}_2(x,x)] - \frac{3\mathsf{p}}{2}\ {}^3\sigma_x({}^2\sigma_x +\ {}^3\sigma_x)\{\mathsf{F}(\frac{(b,x+y>}{2})\ {}_\sigma\int_0^\infty[\mathsf{F}(\frac{(b,2x+\zeta>}{2})\mathsf{K}(x,x+\zeta)]d\zeta\}$.

From Equations $(16,31,33)$ the identity

(34) $\quad (I-A_x)[L_2\mathsf{K}(x,y)] = \{\sigma_\xi[-\frac{3\mathsf{p}}{2}(I-A_x)(\sigma_x +\ {}_1\sigma_y)\mathsf{K}(x,y)$

$-\frac{3\mathsf{p}^2}{4}[(I-A_x)\mathsf{K}(x,y)]\mathsf{K}_2(x,x)]\mathsf{K}_2(\xi,\xi)\}|_{\xi=x}$

$-\frac{3\mathsf{p}}{2}\{\mathsf{F}(\frac{(b,x+y>}{2})(-2\ {}^2\sigma_x)(\mathsf{K}(x,x)\mathsf{K}(x,x)) - 2[({}^2\sigma_x -\ {}^2_1\sigma_y)\mathsf{F}(\frac{(b,x+y>}{2})(\mathsf{K}(x,y)\mathsf{K}(x,x))]|_{y=x}\}$

follows. Expressing F from Equality (1):

$\mathsf{F}(\frac{(b,x+y>}{2}) = \mathsf{F}(x,y) = (I - A_x)\mathsf{K}(x,y)$
and using Formulas (21) and 4(7, 8) and the invertibility of the operator $(I - A_x)$ and 7(13) (see Remark 3.2 also) we get

(35) $L_2\mathsf{K}(x,y) = \{[-\frac{3\mathsf{p}}{2}(\sigma_x + {}_1\sigma_y)\mathsf{K}(x,y) - \frac{3\mathsf{p}^2}{4}\mathsf{K}(x,y)\mathsf{K}_2(x,x)](-\mathsf{K}^2(x,x)\}$
$-\frac{3\mathsf{p}}{2}\{\mathsf{K}(x,y)[(-2\,{}^2\sigma_x)(\mathsf{K}(x,x)\mathsf{K}(x,x))] - 2\mathsf{K}(x,y)[({}^1\sigma_x - {}^1_1\sigma_y)(\mathsf{K}(x,y)\mathsf{K}(x,x))]|_{y=x}\}$
$= 3\mathsf{p}({}^1\sigma_x + {}_1\sigma_y)\mathsf{K}(x,y)\mathsf{K}^2(x,x) + \frac{3\mathsf{p}^2}{2}[\mathsf{K}(x,y)\mathsf{K}_2(x,x)]\mathsf{K}^2(x,x)$
$+3\mathsf{p}\mathsf{K}(x,y)[\,{}^2\sigma_x(\mathsf{K}(x,x)\mathsf{K}(x,x))] + 3\mathsf{p}\mathsf{K}(x,y)[({}^1\sigma_x - {}^1_1\sigma_y)(\mathsf{K}(x,y)\mathsf{K}(x,x))]|_{y=x}.$

Using Formula (22) permits to simplify Formula (35) in the considered case to:

(36) $L_2\mathsf{K}(x,y) = 3\mathsf{p}[(\sigma_x + \sigma_y)\mathsf{K}(x,y)]\mathsf{K}^2(x,x) + 3\mathsf{p}\mathsf{K}(x,y)[{}^2\sigma_x(\mathsf{K}(x,x)\mathsf{K}(x,x))]$.

Putting $g(x,t) = \mathsf{K}(x,x,t)$ one gets particularly for $\sigma = {}_1\sigma$ from Equation (36):

(37) $({}_2\sigma_t + \sigma_x^3)g(x,t) = 3\mathsf{p}[\sigma_x g(x,t)]g^2(x,t) + 3\mathsf{p}g(x,t)[{}^2\sigma_x(g(x,t)g(x,t))]$.

The latter two partial differential equations are non-linear even over the real field. Particularly, if ${}_2\sigma_t = \partial/\partial t_0$ and $\sigma_x = \partial/\partial x_0$ Equation (37) gives the modified Korteweg-de-Vries equation:

$\partial g(x,t)/\partial t + \partial^3 g(x,t)/\partial x^3 - 6pg^2(x,t)\partial g(x,t)/\partial x = 0$ with $t = t_0$ and $x = x_0$.

Find particular solutions of (36), when
${}_2\sigma_t f = (\partial f/\partial t_0)$ and $\sigma_x g(x) = \sum_{n_0+1}^{n+n_0} i_j^*[\partial g(x)/\partial x_j]$, using
$\mathsf{F}(\frac{(b,x+y>}{2}), t) := (nt)^{-1/3} f(\xi_{n_0+1} + \eta_{n_0+1}, ..., \xi_{n+n_0} + \eta_{n+n_0})$ with $\xi = x(nt)^{-1/3}$ and $\eta = y(nt)^{-1/3}$, $-1 \le n_0$, $2 \le n$, $n_0 + n \le 3$, $t = t_0$. While with the help of the operators

$L_{2,j}g(x,y,t) = \frac{1}{n}i_j^*\partial_t g(x,y,t) + (\partial_{x_j}\sigma_x^2 + \partial_{y_j}\sigma_y^2 + 3\partial_{x_j}\sigma_y^2 + 3\partial_{y_j}\sigma_x^2)g(x,y,t)$
and the conditions $L_{2,j}F = 0$ for each $j = n_0 + 1, ..., n_0 + n$ one obtains the partial differential equations

$f + \xi_j f + \partial_{\xi_j} \sum_{n_0+1}^{n+n_0} \partial^2_{\xi_k} f = 0$
for each $j = n_0 + 1, ..., n + n_0$, where $f = f(\xi_{n_0+1}, ..., \xi_{n_0+n})$, $\partial_{x_j} = \partial/\partial_{x_j}$. Solving these partial differential equations we get the solution $\mathsf{K}(x,y) = (nt)^{-1/3}\kappa(\xi_{n_0+1}, ..., \xi_{n_0+n}, \eta_0, ..., \eta_{n_0+n})$ from the integral equation

$\hat{K}(\xi,\eta) = \hat{F}(\xi+\eta) + \frac{p}{4}\,(\,{}_\sigma\int_\xi^\infty(\,{}_\sigma\int_\xi^\infty \hat{F}(u+\eta)[\hat{F}(z+u)\hat{K}(\xi,z)]dz)du$,
where $\hat{K}(\xi,\eta) = \kappa(\xi_{n_0+1}, ..., \xi_{n_0+n}, \eta_0, ..., \eta_{n_0+n})$, $\xi = \xi_0 i_0 + ... + \xi_3 i_3$.

Each solution of the modified Korteweg-de-Vries equation generates a solution of Kadomtzev-Petviashvili partial differential equation due to Miura's transform: $g = -v^2 - \sigma_x v$. In the considered non-commutative case it takes the form:

${}_2\sigma_t(v^2) + {}_2\sigma_t\sigma_x v + (\sigma_x^3(v^2) + \sigma_x^4 v) - 3\mathsf{p}(\sigma_x(v^2) + \sigma_x^2 v)(v^4 + v^2\sigma_x v + (\sigma_x v)v^2 + (\sigma_x v)^2) - 3\mathsf{p}(v^2 + \sigma_x v)[{}^2\sigma_x((v^2 + \sigma_x v)(v^2 + \sigma_x v))]$.

Thus we have demonstrated the following statement.

9.1 Theorem. *Partial differential Equation* (36) *with* $\psi_0 = {}_1\psi_0 = 0$ *over the Cayley-Dickson algebra* $\mathcal{A}_r$, $2 \le r \le 3$, *has a solution given by Formulas* $(1-3, 25, 26)$, *when the appearing integrals uniformly converge by parameters as in Proposition 4 and the operator* $(I - A_x)$ *is invertible and* $\mathsf{F} \in Mat_s(\mathbf{R})$ *and* $\mathsf{K} \in Mat_s(\mathcal{A}_r)$ *with* $s \in \mathbf{N}$ *for* $r = 2$ *and* $s = 1$ *for* $r = 3$.

10. Example. Now we consider the pair of partial differential operators

(1) $L_1 = \sigma_x + \sigma_y$ and

(2) $L_2 = {}_1\sigma_t + \sigma_x^2 + u\sigma_y\sigma_x + \sigma_y^2$,

where $u \in \mathbf{R}$ is a real constant, and the integral equation:

(3) $\mathsf{K}(x,y) = \mathsf{F}(x,y) + \mathsf{p}\ {}_\sigma\int_x^\infty \mathsf{F}(z,y)\mathsf{K}(x,z)dz$.

Acting on the both sides of the latter equation with the operator L_2 and using the condition

(5) $L_{2,j}\mathsf{F}(x,y) = 0$ for each j, where

(6) $L_{2,j}\mathsf{F}(x,y) = [{}_1\psi_j\partial_{t_j} +$

$u\sum_{s,p:\ i_si_p=i_j} \psi_s\psi_p(\partial_{y_s}\partial_{x_p} + (i_pi_s)(i_si_p)\partial_{y_p}\partial_{x_s}) + \pi_j \circ (\sigma_x^2 + \sigma_y^2)]\mathsf{F}(x,y)$,

where $\pi_j : X \to X_j$ is the $\mathbf{R}$ linear projection operator (see §1.2). We infer that

(7) $L_2\mathsf{K}(x,y) = \mathsf{p}({}_1^1\sigma_t + \sigma_x^2 + u\sigma_y\sigma_x + \sigma_y^2 + {}_1^2\sigma_t)\ {}_\sigma\int_x^\infty \mathsf{F}(z,y)\mathsf{K}(x,z)dz$

$= \mathsf{p}({}_1^2\sigma_t + \sigma_x^2 + u\ {}^1\sigma_y\sigma_x -\ {}^1\sigma_z^2 - u\ {}^1\sigma_y\ {}^1\sigma_z)\ {}_\sigma\int_x^\infty \mathsf{F}(z,y)\mathsf{K}(x,z)dz$,

since $({}_1\sigma_t + \sigma_y^2)\mathsf{F}(z,y) = -(\sigma_z^2 + u\sigma_y\sigma_z)\mathsf{F}(z,y)$. In view of Proposition 4 and Corollary 5 we get the formula:

(8) $L_2\mathsf{K}(x,y) = I_1 + I_2$,

where

(9) $I_1 = \mathsf{p}(\ {}_1^2\sigma_t + \sigma_x^2 + u\ {}^1\sigma_y\ \sigma_x)\ {}_\sigma\int_x^\infty \mathsf{F}(z,y)\mathsf{K}(x,z)dz$

$= \mathsf{p}({}^2\sigma_x^2 + u\ {}^1\sigma_y\ {}^2\sigma_x)\ {}_\sigma\int_x^\infty \mathsf{F}(z,y)\mathsf{K}(x,z)dz$

$-\mathsf{p}\sigma_x[\mathsf{F}(x,y)\mathsf{K}(x,x)] - \mathsf{p}\ {}^2\sigma_x[\mathsf{F}(x,y)\mathsf{K}(x,z)]|_{z=x} - u\mathsf{p}\sigma_y[\mathsf{F}(x,y)\mathsf{K}(x,x)]$ and

(10) $\quad -I_2 = \mathsf{p}({}^1\sigma_z^2 + u\ {}^1\sigma_y\ {}^1\sigma_z)\ {}_\sigma\int_x^\infty \mathsf{F}(z,y)\mathsf{K}(x,z)dz$

$= \mathsf{p}({}^2\sigma_z^2 - u\ {}^1\sigma_y\ {}^2\sigma_z)\ {}_\sigma\int_x^\infty \mathsf{F}(z,y)\mathsf{K}(x,z)dz$

$-\mathsf{p}\ {}^1\sigma_x[\mathsf{F}(x,y)\mathsf{K}(x,x)] + \mathsf{p}\ {}^2\sigma_z[\mathsf{F}(x,y)\mathsf{K}(x,z)]|_{z=x} - u\mathsf{p}\sigma_y[\mathsf{F}(x,y)\mathsf{K}(x,x)]$.

Then using the second condition

(11) $L_1\mathsf{F}(x,y) = 0$, i.e. $\mathsf{F}(x,y) = \mathsf{F}(\frac{(\psi, x-y>}{2})$,

it is possible to simplify certain terms appeared above:

(12) $\sigma_y\ {}^2\sigma_x\ {}_\sigma\int_x^\infty \mathsf{F}(z,y)\mathsf{K}(x,z)dz$

$=\ {}^2\sigma_z\ {}^2\sigma_x\ {}_\sigma\int_x^\infty \mathsf{F}(z,y)\mathsf{K}(x,z)dz +\ {}^2\sigma_x[\mathsf{F}(x,y)\mathsf{K}(x,z)]|_{z=x}$ and

(13) ${}^1\sigma_y\ {}^2\sigma_z\ {}_\sigma\int_x^\infty \mathsf{F}(z,y)\mathsf{K}(x,z)dz = -\ {}^1\sigma_z\ {}^2\sigma_z\ {}_\sigma\int_x^\infty \mathsf{F}(z,y)\mathsf{K}(x,z)dz$

$=\ {}^2\sigma_z^2\ {}_\sigma\int_x^\infty \mathsf{F}(z,y)\mathsf{K}(x,z)dz +\ {}^2\sigma_z[\mathsf{F}(x,y)\mathsf{K}(x,z)]|_{z=x}$.

Therefore, Equations $(7-13)$ lead to the identity:

(14) $L_2\mathsf{K}(x,y) = \mathsf{p}({}_1^2\sigma_t + {}^2\sigma_x^2 + u\ {}^2\sigma_z\ {}^2\sigma_x + (u-1)\ {}^2\sigma_z^2)\ {}_\sigma\int_x^\infty \mathsf{F}(z,y)\mathsf{K}(x,z)dz$

$-(u+2)\mathsf{p}\ {}^2\sigma_x[\mathsf{F}(x,y)\mathsf{K}(x,x)] + u\mathsf{p}\ {}^2\sigma_x[\mathsf{F}(x,y)\mathsf{K}(x,z)]|_{z=x} + u\mathsf{p}\ {}^2\sigma_z[\mathsf{F}(x,y)\mathsf{K}(x,z)]|_{z=x}$.

Take $u = 2$. If $\mathsf{F}(x,y) \in Mat_s(\mathbf{R})$ and $\mathsf{K}(x,y) \in Mat_s(\mathcal{A}_r)$ for each $x, y \in U$, $2 \le r \le 3$, with $s \in \mathbf{N}$ for $r = 2$ and $s = 1$ for $r = 3$, then Formulas $(3,14)$ and $2(7)$ and $7(2,13)$ and $5(7)$ for the invertible operator $(I - A_x)$ imply, that

(15) $({}_1\sigma_t + \sigma_x^2 + 2\ {}^2\sigma_y\ {}^2\sigma_x + \sigma_y^2)\mathsf{K}(x,y) = -2\mathsf{p}\mathsf{K}(x,y)[\sigma_x \mathsf{K}(x,x)]$.

Putting $g(x,t) = \mathsf{K}(x,x)$ we get particularly on the diagonal $x = y$:

(16) $({}_1\sigma_t + \sigma_x^2)g(x,t) = -2\mathsf{p}g(x,t)[\sigma_x g(x,t)]$.

If $u \in \mathcal{I}_r := \{z \in \mathcal{A}_r :\ Re(z) = 0\}$ and $\psi_0 = 0$, then ${}^2\sigma_x(u(x,t)u(x,t)) + u(x,t)(\sigma_x u(x,t)) = \sum_{j>0} u_j(x,t)(\partial u(x,t)/\partial x_j)\psi_j$.

This approach has the following application.

A non-isothermal flow of a non-compressible Newtonian liquid with a dissipative heating is described by the system of partial differential equations:

(17) $\rho \sum_{j=1}^3 u_j \partial u/\partial x_j = -\frac{1}{2}\mu\sigma_x^2 u + \rho\frac{\partial u}{\partial t_0}$

(18) $div(u) = \sum_{j=1}^3 \partial u_j/\partial x_j = 0$

(19) $\rho c_p \sum_{j=1}^3 u_j \partial T/\partial x_j = -\lambda\sigma_x^2 T + 2\mu I_2$,

when a density ρ and a dynamical viscosity μ are independent of coordinates x_1, x_2, x_3, where

$I_2 = \frac{1}{4}\sum_{j,k=1}^3(\partial u_k/\partial x_j + \partial u_j/\partial x_k)^2$

denotes the tensor of deformation velocities, T is the temperature, also λ and c_p are physical constants. Here we take $u = u_1 i_1 + u_2 i_2 + u_3 i_3$ with real-valued functions u_1, u_2, u_3 and $\sigma_x f(x) = \sum_{j=1}^3 i_j^*(\partial f(x)/\partial x_j)$, $x = x_0 i_0 + x_1 i_1 + x_2 i_2 + ... + x_{2^r-1} i_{2^r-1} \in \mathcal{A}_r$, $r = 2$ or $r = 3$, x_0 and x_j with $j > 3$ can be taken constant, for example, $x_0 = 0$, $x_j \in \mathbf{R}$ for each j, ${}_1\sigma_t = {}_1\psi_0\ \partial_{t_0}$ and choose suitable constants ${}_1\psi_0$ and p.

For usual liquids like water the dependence of the dynamical viscosity on the temperature is described as $\mu = \mu_0(T_0/T)^m$, where $m \ge 0$. For very viscous Newtonian liquids like glycerin the function $\mu = \mu_0\exp[-\beta(T - T_0)]$ is usually taken with empirical constants μ_0, T_0 and β (see also [18]). Using the method described above it is possible to find solutions satisfying Equation (20) and take into account Condition (21) with the help of $\mathsf{F}(x,x)$ and then $\mathsf{K}(x,x)$ satisfying it.

Put $g = v + u$, where $u \in \mathcal{I}_2 := \{z \in \mathcal{A}_2 :\ Re(z) = 0\}$ and $v \in \mathcal{A}_r \ominus \mathcal{I}_2$, where $2 \le r \le 3$. If g is a solution of partial differential equation (16) and

$v(\sigma u) + u(\sigma v) =\ {}^2\sigma(uu + vv)$,

then u is a solution of partial differential equation (17). Condition (18) means that $Re(\sigma u) = 0$ or $\sigma u + (\sigma u)^* = 0$. Calculating I_2 and solving Equation (19) one also calculates the temperature.

The result of this section can be formulates as the following theorem.

10.1 Theorem. *Partial differential Equation* (15) *over the Cayley-Dickson*

algebra $\mathcal{A}_r$ with $2 \le r \le 3$ has a solution given by Formulas $(1, 3, 5, 6, 11)$, *when the appearing integrals uniformly converge by parameters on each compact sub-domain (see Proposition 4) and the operator* $(I - A_x)$ *is invertible and* $\mathsf{F} \in Mat_s(\mathbf{R})$ *and* $\mathsf{K} \in Mat_s(\mathcal{A}_r)$, $s \in \mathbf{N}$ *for* $r = 2$, $s = 1$ *for* $r = 3$.

11. Remark. The results presented above show that the method of commutators of integral operators with partial differential operators over complex numbers becomes more powerful with the use of Cayley-Dickson algebras and the non-commutative line integral over them and permits to solve more general non-linear partial differential equations. It is planned to develop further this method for partial differential equations with variable coefficients and also for generalized functions and equations with generalized coefficients.

References

[1] M. J. Ablowitz, H. Segur. Solitons and the inverse scattering transform (SIAM: Philadeplhia, 1981).

[2] J.C. Baez. The octonions. // Bull. Amer. Mathem. Soc. **39: 2**, 145-205 (2002).

[3] R.V. Cadison, J.R. Ringrose. Fundamentals of the theory of operator algebras. V. 1,2 (Academic Press: New York, 1983).

[4] G. Emch. Méchanique quantique quaternionienne et Relativitè restreinte. // Helv. Phys. Acta **36**, 739-788 (1963).

[5] F. Gürsey, C.-H. Tze. On the role of division, Jordan and related algebras in particle physics (World Scientific Publ. Co.: Singapore, 1996).

[6] L.I. Kamynin. Course of Mathematical Analysis (Moscow State Univ. Press: Moscow, 1995).

[7] I.L. Kantor, A.S. Solodovnikov. Hypercomplex numbers (Springer-Verlag: Berlin, 1989).

[8] A.G. Kurosh. Lectures on general algebra (Nauka: Moscow, 1973).

[9] S.V. Ludkovsky. Differentiable functions of Cayley-Dickson numbers and line integration. // J. Mathem. Sciences **141: 3**, 1231-1298 (2007).

[10] S.V. Lüdkovsky, F. van Oystaeyen. Differentiable functions of quaternion variables.// Bull. Sci. Math. **127**, 755-796 (2003).

[11] S.V. Ludkovsky. Analysis over Cayley-Dickson numbers and its applications (Lambert Academic Publishing: Saarbrücken, 2010).

[12] S.V. Ludkovsky. Differential equations over octonions.// Adv. Appl. Clifford Alg., DOI: 10.1007/s00006-011-0286-4 Online First, in press 2011 (previous variant: Los Alamos Nat. Lab. math/1003.2620, 50 pages).

[13] S.V. Ludkovsky. Feynman integration over octonions with application to quantum mechanics. // Mathematical Methods in the Appl. Sciences **33: 9** (2010), 1148-1173.

[14] S.V. Ludkovsky, W. Sproessig. Ordered representations of normal and super-differential operators in quaternion and octonion Hilbert spaces.// Adv. Appl. Clifford Alg., **20: 2** (2010), 321-342.

[15] S.V. Ludkovsky. Algebras of operators in Banach spaces over the quaternion skew field and the octonion algebra.// J. Mathem. Sci. **144: 4** (2008), 4301-4366.

[16] S.V. Ludkovsky. Line integration and second order partial differential equations over Cayley-Dickson algebras. Los Alam. Nat. Lab. **math.CV:1009.2158**, 80 pages.

[17] A.D. Polyanin, V.F. Zaytzev, A.I. Jurov. Methods of solutions of non-linear equations of mathematical physics and mechanics (Fizmatlit: Moscow, 2005).

[18] A.D. Polyanin, V.F. Zaytzev. Reference book on non-linear equations of mathematical physics (Fizmatlit: Moscow, 2002).

[19] A.G. Sveshnikov, A.B. Alshin, M.O. Korpusov, Yu.D. Pletner. Linear and non-linear equations of Sobolev type (Fizmatlit: Moscow, 2007).

Department of Applied Mathematics,
Moscow State Technical University MIREA, av. Vernadsky 78,
Moscow, Russia
e-mail: sludkowski@mail.ru